Energy- and flux-budget (EFB) turbulence closure model for stably stratified flows. Part I: steady-state, homogeneous regimes

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Abstract We propose a new turbulence closure model based on the budget equations for the key second moments: turbulent kinetic and potential energies: TKE and TPE (comprising the turbulent total energy: TTE = TKE + TPE) and vertical turbulent fluxes of momentum and buoyancy (proportional to potential temperature). Besides the concept of TTE, we take into account the non-gradient correction to the traditional buoyancy flux formulation. The proposed model permits the existence of turbulence at any gradient Richardson number, Ri. Instead of the critical value of Richardson number separating—as is usually assumed—the turbulent and the laminar regimes, the suggested model reveals a transitional interval, $0.1 < \mathrm{Ri} < 1$, which separates two regimes of essentially different nature but both turbulent: strong turbulence at $\mathrm{Ri} \ll 1$; and weak turbulence, capable of transporting momentum but much less efficient in transporting heat, at $\mathrm{Ri} > 1$. Predictions from this model are consistent with available data from atmospheric and laboratory experiments, direct numerical simulation and large-eddy simulation.

Keywords Anisotropy · Critical Richardson number · Eddy viscosity · Heat conductivity · Kinetic, potential and total turbulent energies · Stable stratification · Turbulence closure · Turbulent fluxes · Turbulent length scale

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1 Introduction

Most of practically-used turbulence closure models are based on the concept of downgradient transport. Accordingly the models express turbulent fluxes of momentum and scalars as products of the mean gradient of the transported property and the corresponding turbulent transport coefficient (eddy viscosity, K_M, heat conductivity, K_H, or diffusivity, K_D). Following Kolmogorov (1941), turbulent transport coefficients are taken to be proportional to the turbulent velocity scale, u_T, and length scale, l_T:

$$K_M \sim K_H \sim K_D \sim u_T l_T . \tag{1}$$

Usually u_T^2 is identified with the turbulent kinetic energy (TKE) per unit mass, E_K, and is calculated from the TKE budget equation using the Kolmogorov closure for the TKE dissipation rate:

$$\varepsilon_K \sim E_K/t_T , \tag{2}$$

where $t_T \sim l_T/u_T$ is the turbulent dissipation time scale. This approach is justified when it is applied to neutral stability flows, where l_T can be taken to be proportional to the distance from the nearest wall.

However, this method encounters difficulties in stratified flows (both stable and unstable). The turbulent Prandtl number $\mathrm{Pr}_T = K_M/K_H$ exhibits essential dependence on the stratification and cannot be considered as constant. Furthermore, as follows from the budget equations for the vertical turbulent fluxes, the velocity scale u_T, which characterises the vertical turbulent transport, is determined as the root mean square (r.m.s.) vertical velocity $u_T \sim \sqrt{E_z}$ (where E_z is the energy of the vertical velocity fluctuations). In neutral stratification $E_z \sim E_K$, which is why the traditional equation $u_T \sim \sqrt{E_K}$ holds true. However, in strongly stable stratification this equation is insufficiently accurate because of the stability dependence of the anisotropy of turbulence $A_z \equiv E_z/E_K$, e.g., A_z generally decreases with increasing stability.

To reflect the effect of stratification, the turbulent length scales for the momentum, l_{TM}, and heat, l_{TH}, are taken to be unequal. As a result, the above-described closure scheme (formulated by Kolmogorov for neutral stratification and well-grounded only in this case) loses its constructiveness: the unsolved part of the problem is merely displaced from $\{K_M, K_H\}$ to $\{l_{TM}, l_{TH}\}$. In that case, the TKE budget equation becomes insufficient to determine additional unknown parameters.

Numerous alternative turbulence closures have been formulated using the budget equations for other turbulent parameters (in addition to the TKE) together with heuristic hypotheses and empirical relationships. However no consensus has been reached (see overviews by Weng and Taylor 2003; Umlauf and Burchard 2005).

In this study we analyse the effects of density stratification on turbulent energies and vertical turbulent fluxes in stably stratified atmospheric (or oceanic) boundary layers, in which the horizontal variations of the mean velocity and temperature are much weaker than the vertical variations. The proposed theory provides realistic stability dependencies of the turbulent Prandtl number, the vertical anisotropy, and the vertical turbulent length scale. Our work is presented in meteorological terms, but all the results can be easily reformulated in terms of water currents in oceans or lakes. In this case buoyancy is expressed through temperature and salinity instead of temperature and humidity.

We consider a minimal set of the budget equations for the second-order moments, namely equations for the vertical fluxes of buoyancy (proportional to the potential temperature) and momentum, the TKE and the turbulent potential energy (TPE), proportional to the mean squared potential temperature fluctuation. In these equations we account for some commonly neglected effects but leave a more detailed treatment of the third-order transports and the pressure–velocity correlations for future analysis. In particular, we advance the familiar "return to isotropy" model in order to more realistically determine the stability dependence of the vertical anisotropy, A_z. We also take into account a non-gradient correction to the traditional, downgradient formulation for the turbulent flux of potential temperature. This approach allows us to derive a reasonably simple turbulence closure scheme including realistic energy budgets and stability dependence of Pr_T.

We consider the total (kinetic + potential) turbulent energy (TTE), derive the TTE budget equation, and demonstrate that the TTE in stably stratified sheared flows does not completely decay even in very strong static stability. This conclusion, which is deduced from the general equations independently of the concrete formulation for the turbulent length scale, refutes the widely accepted concept of the critical Richardson number.

For the reader's convenience we recall that the gradient Richardson number, Ri, is defined as the squared ratio of the Brunt–Väisälä frequency, N, to the velocity shear, S:

$$\mathrm{Ri} = \left(\frac{N}{S}\right)^2, \tag{3a}$$

$$S^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2, \tag{3b}$$

$$N^2 = \beta \frac{\partial \Theta}{\partial z}, \tag{3c}$$

where z is the vertical coordinate, U and V are the mean velocity components along the horizontal axes x and y, Θ is the mean potential temperature, $\beta = g/T_0$ is the buoyancy parameter, $g = 9.81\,\mathrm{m\,s^{-1}}$ is the acceleration due to gravity, and T_0 is a reference absolute temperature. As originally proposed by Richardson (1920), the Richardson number quantifies the effect of static stability on turbulence. Subsequent researches in the theory of stably stratified turbulent flows focussed on the question whether or not stationary turbulence can be maintained by the velocity shear at very large Richardson numbers.

A widely accepted opinion is that turbulence decays when Ri exceeds some critical value, $\mathrm{Ri_c}$ (with the frequently quoted estimate of $\mathrm{Ri_c} = 0.25$). However, the concept of a critical Ri was neither rigorously derived from basic physical principles nor demonstrated empirically; indeed, it contradicts long standing experimental evidence.

It is worth emphasizing that turbulence closure models based on the straightforward application of the TKE budget equation and Kolmogorov's closure hypotheses, Eqs. (1) and (2), imply the existence of Ri_c. In practical atmospheric modelling these closures are not acceptable. In particular, they lead to unrealistic decoupling of the atmosphere from the underlying surface when the Richardson number in the surface layer exceeds Ri_c. Since the milestone study of Mellor and Yamada (1974), in order to prevent the undesirable appearance of Ri_c, turbulence closures in practical use have been equipped with correction coefficients specifying the ratios $K_M(u_T l_T)^{-1}$ and $K_H(u_T l_T)^{-1}$ in the form of different single-valued functions of Ri. Usually these functions are not derived in the context of the closure in use, but are either determined empirically or taken from independent theories. Using this approach, modellers ignore the fact that corrections could be inconsistent with the formalism of the basic closure model.

2 Reynolds equations and budget equations for second moments

We consider atmospheric flows in which typical variations of the mean wind velocity $\mathbf{U} = (U_1, U_2, U_3) = (U, V, W)$ and potential temperature Θ (or virtual potential temperature involving specific humidity) in the vertical direction [along the x_3 (or z) axis] are much larger than in the horizontal direction [along the x_1, x_2 (or x, y) axes], so that the terms proportional to their horizontal gradients in the budget equations for turbulent statistics can be neglected. Θ is defined as $\Theta = T(P_0/P)^{1-1/\gamma}$, where T is the absolute temperature, P is the pressure, P_0 is its reference value, and $\gamma = c_p/c_v = 1.41$ is the ratio of specific heats.

We also assume that the vertical scale of motions (which is limited to the height scale of the atmosphere or the ocean, $H \sim 10^4$ m) is much smaller than the horizontal scale, so that the mean flow vertical velocity is typically much smaller than the horizontal velocity. In this context, to close the Reynolds equations we need only the vertical component, F_z, of the potential temperature flux, F_i, and two components of the Reynolds stresses, τ_{ij}, that represent the vertical turbulent flux of momentum: τ_{13} and τ_{23}.

The mean flow is determined by the momentum equations:

$$\frac{\mathrm{D}U_1}{\mathrm{D}t} = fU_2 - \frac{1}{\rho_0}\frac{\partial P}{\partial x} - \frac{\partial \tau_{13}}{\partial z}, \tag{4}$$

$$\frac{\mathrm{D}U_2}{\mathrm{D}t} = -fU_1 - \frac{1}{\rho_0}\frac{\partial P}{\partial y} - \frac{\partial \tau_{23}}{\partial z}, \tag{5}$$

and the thermodynamic energy equation:

$$\frac{\mathrm{D}\Theta}{\mathrm{D}t} = -\frac{\partial F_z}{\partial z} + J, \tag{6}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U_k \partial/\partial x_k$, $\tau_{ij} = \langle u_i u_j \rangle$, $F_i = \langle u_i \theta \rangle$, t is the time, $f = 2\Omega \sin\varphi$, with Ω_i the earth's rotation vector parallel to the polar axis ($|\Omega_i| \equiv \Omega = 0.76 \times 10^{-4}\,\mathrm{s}^{-1}$), φ is the latitude, ρ_0 is the mean density, J is the heating/cooling rate ($J = 0$ in adiabatic processes), P is the mean pressure, $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ and θ are the velocity and potential–temperature fluctuations. The angle brackets denote the ensemble average [see Holton (2004) or Kraus and Businger (1994)].

The budget equations for the TKE, $E_K = \frac{1}{2}\langle u_i u_i \rangle$, the "energy" of the potential temperature fluctuations, $E_\theta = \frac{1}{2}\langle \theta^2 \rangle$, the potential temperature flux, $F_i = \langle u_i \theta \rangle$ [with the vertical component $F_3 = F_z = \langle w\theta \rangle$], and the Reynolds stress $\tau_{ij} = \langle u_i u_j \rangle$ [with the components $\tau_{i3} = \langle u_i w \rangle$ ($i = 1, 2$) representing the vertical flux of momentum] read (see, e.g., Kaimal and Finnigan (1994), Kurbatsky (2000) and Cheng et al. (2002)):

$$\frac{\mathrm{D}E_K}{\mathrm{D}t} + \nabla \cdot \boldsymbol{\Phi}_K = -\tau_{ij}\frac{\partial U_i}{\partial x_j} + \beta F_z - \varepsilon_K \tag{7a}$$

or approximately

$$\frac{\mathrm{D}E_K}{\mathrm{D}t} + \frac{\partial \Phi_K}{\partial z} \approx -\tau_{i3}\frac{\partial U_i}{\partial z} + \beta F_z - \varepsilon_K, \tag{7b}$$

$$\frac{\mathrm{D}E_\theta}{\mathrm{D}t} + \nabla \cdot \boldsymbol{\Phi}_\theta = -F_z\frac{\partial \Theta}{\partial z} - \varepsilon_\theta, \tag{8a}$$

or approximately

$$\frac{DE_\theta}{Dt} + \frac{\partial \Phi_\theta}{\partial z} = -F_z \frac{\partial \Theta}{\partial z} - \varepsilon_\theta, \tag{8b}$$

$$\frac{DF_i}{Dt} + \frac{\partial}{\partial x_j}\Phi_{ij}^{(F)} = \beta_i \left\langle \theta^2 \right\rangle + \frac{1}{\rho_0} \left\langle \theta \nabla_i p \right\rangle - \tau_{ij} \frac{\partial \Theta}{\partial z}\delta_{j3} - F_j \frac{\partial U_i}{\partial x_j} - \varepsilon_i^{(F)}, \tag{9a}$$

and for $F_3 = F_z$

$$\begin{aligned} \tfrac{DF_z}{Dt} + \tfrac{\partial}{\partial z}\Phi_F &= \beta \left\langle \theta^2 \right\rangle + \tfrac{1}{\rho_0} \left\langle \theta \tfrac{\partial}{\partial z} p \right\rangle - \left\langle w^2 \right\rangle \tfrac{\partial \Theta}{\partial z} - \varepsilon_z^{(F)} \\ &\approx C_\theta \beta \left\langle \theta^2 \right\rangle - \left\langle w^2 \right\rangle \tfrac{\partial \Theta}{\partial z} - \varepsilon_z^{(F)}, \end{aligned} \tag{9b}$$

$$\frac{D\tau_{ij}}{Dt} + \frac{\partial}{\partial x_k}\Phi_{ijk}^{(\tau)} = -\tau_{ik}\frac{\partial U_j}{\partial x_k} - \tau_{jk}\frac{\partial U_i}{\partial x_k} + \left[\beta(F_j \delta_{i3} + F_i \delta_{j3}) + Q_{ij} - \varepsilon_{ij}^{(\tau)}\right] \tag{10a}$$

and for τ_{i3} (i = 1, 2)

$$\frac{D\tau_{i3}}{Dt} + \frac{\partial}{\partial z}\Phi_i^{(\tau)} = -\left\langle w^2 \right\rangle \frac{\partial U_i}{\partial z} - \left[-\beta F_i - Q_{i3} + \varepsilon_{i3}^{(\tau)}\right] \approx -\langle w^2 \rangle \frac{\partial U_i}{\partial z} - \varepsilon_{i3}, \tag{10b}$$

where $\beta_i = \beta e_i$ and $\mathbf{e}$ is the vertical unit vector, $F_i = \langle u_i \, \theta \rangle$ $(i = 1, 2)$ are the horizontal fluxes of potential temperature, $-\tau_{ij}\partial U_i/\partial x_j$ is the TKE production rate, and δ_{ij} is the unit tensor (δ_{ij}= 1 for $i = j$ and δ_{ij}= 0 for $i \neq j$).

Here, $\mathbf{\Phi}_K$, $\mathbf{\Phi}_\theta$, etc. are the third-order moments representing the turbulent transports of the TKE and the "energy" of potential temperature fluctuations:

$$\mathbf{\Phi}_K = \tfrac{1}{\rho_0} \langle p \, \mathbf{u} \rangle + \tfrac{1}{2} \left\langle u^2 \, \mathbf{u} \right\rangle, \quad \text{that is } \Phi_K = \tfrac{1}{\rho_0} \langle p \, w \rangle + \tfrac{1}{2} \left\langle u^2 \, w \right\rangle, \tag{11a}$$

$$\mathbf{\Phi}_\theta = \tfrac{1}{2} \left\langle \theta^2 \, \mathbf{u} \right\rangle, \quad \text{that is } \Phi_\theta = \tfrac{1}{2} \left\langle \theta^2 \, w \right\rangle, \tag{11b}$$

and the turbulent transports of the fluxes of potential temperature and momentum:

$$\Phi_{ij}^{(F)} = \frac{1}{2\rho_0} \langle p \, \theta \rangle \, \delta_{ij} + \left\langle u_i \, u_j \, \theta \right\rangle, \quad \Phi_{33}^{(F)} = \Phi_F = \frac{1}{2\rho_0} \langle p \, \theta \rangle + \left\langle w^2 \, \theta \right\rangle, \tag{12}$$

$$\Phi_{ijk}^{(\tau)} = \left\langle u_i u_j u_k \right\rangle + \frac{1}{\rho_0} \left(\langle p u_i \rangle \, \delta_{jk} + \left\langle p u_j \right\rangle \delta_{ik} \right), \tag{13a}$$

$$\Phi_{i33}^{(\tau)} = \Phi_i^{(\tau)} = \left\langle u_i w^2 \right\rangle + \frac{1}{\rho_0} \langle p u_i \rangle, \, (i = 1, 2). \tag{13b}$$

Q_{ij} are correlations between the fluctuations of pressure, p, and the velocity shears:

$$Q_{ij} = \frac{1}{\rho_0} \left\langle p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle. \tag{14}$$

In the above equations, ε_K, $\varepsilon_{ij}^{(\tau)}$, ε_θ and $\varepsilon_i^{(F)}$ are operators including the molecular transport coefficients:

$$\varepsilon_K = \nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right\rangle, \quad \varepsilon_{ij}^{(\tau)} = 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle, \tag{15a}$$

$$\varepsilon_\theta = -\kappa \left\langle \theta \, \Delta \, \theta \right\rangle, \quad \varepsilon_i^{(F)} = -\kappa \left(\langle u_i \, \Delta \, \theta \rangle + \Pr \, \langle \theta \, \Delta \, u_i \rangle \right), \tag{15b}$$

where ν is the kinematic viscosity, κ is the temperature conductivity, and $\Pr = \nu/\kappa$ is the Prandtl number. Of these terms, ε_K, $\varepsilon_{ii}^{(\tau)}$ (that is the diagonal elements $\varepsilon_{11}^{(\tau)}$, $\varepsilon_{22}^{(\tau)}$, $\varepsilon_{33}^{(\tau)}$), ε_θ and $\varepsilon_i^{(F)}$ are essentially positive and represent the dissipation rates for E_K, τ_{ii}, E_θ and $F_i^{(F)}$, respectively. Following Kolmogorov (1941), they are taken to be proportional to the ratios of the dissipating statistical moment to the turbulent dissipation time scale, t_T:

$$\varepsilon_K = \frac{E_K}{C_K t_T}, \quad \varepsilon_{ii}^{(\tau)} = \frac{\tau_{ii}}{C_K t_T}, \quad \varepsilon_\theta = \frac{E_\theta}{C_P t_T}, \quad \varepsilon_i^{(F)} = \frac{F_i}{C_F t_T}, \tag{16}$$

where C_K, C_P and C_F are dimensionless constants.

The physical mechanisms of dissipation of the non-diagonal components of the Reynolds stress, $\tau_{ij} (i \neq j)$, are more complicated. The terms $\varepsilon_{ij}^{(\tau)} = 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle$ in Eq. (10b) are comparatively small and are not even necessarily positive, whereas the dissipative role is to a large extent performed by the pressure-shear correlations and the horizontal turbulent transport of the potential temperature. Moreover, our analysis does not account for the vertical transport of momentum (that is for the contribution to τ_{i3}) due to internal gravity waves [see, e.g., Sect. 9.4 in Holton (2004)]. Leaving the detailed analyses of the τ_{i3} budget for future work, we now introduce the following "effective dissipation rate" for the Reynolds stress:

$$\varepsilon_{i3(\mathrm{eff})} \equiv \varepsilon_{i3}^{(\tau)} - \beta F_i - Q_{i3} + (\text{unaccounted factors}), \quad i = 1, 2; \tag{17}$$

and apply to it the Kolmogorov closure hypothesis whereby $\varepsilon_{i3(\mathrm{eff})} \sim \tau_{i3}/t_\tau$, and t_τ is an "effective dissipation time scale" [the term $\varepsilon_{i3}^{(\tau)}$ is estimated as $\varepsilon_{i3}^{(\tau)} \sim O(Re^{-1/2})$ and can be neglected]. Accounting for the difference between t_τ and the Kolmogorov dissipation time scale, t_T [see Eq. (16)], our effective dissipation rates become

$$\varepsilon_{i3(\mathrm{eff})} = \frac{\tau_{i3}}{\Psi_\tau t_T}, \tag{18}$$

where $\Psi_\tau = t_\tau / t_T$ is an empirical dimensionless coefficient. There are no grounds *a prior* to assume that this coefficient is constant. Coefficient Ψ_τ can depend on the static stability but is neither zero nor infinite, and it is also conceivable that this stability dependence is monotonic.

In further analysis we employ the approximate version of Eq. (9b). As shown in Appendix A, the second term on the r.h.s. of Eq. (9b), namely $\rho_0^{-1} \langle \theta \partial p / \partial z \rangle$, is essentially negative and scales as $\beta \langle \theta^2 \rangle$. On these grounds, in its approximate version the sum $\beta \langle \theta^2 \rangle + \rho_0^{-1} \langle \theta \partial p / \partial z \rangle$ is replaced by $C_\theta \beta \langle \theta^2 \rangle$, where $C_\theta < 1$ is an empirical dimensionless constant.

3 Turbulent energies

We first consider the concept of turbulent potential energy (TPE). Using the state equation and the hydrostatic equation, the density and the buoyancy in the atmosphere are expressed through potential temperature, θ, and specific humidity, q (in the ocean, through θ and salinity, s). These variables are adiabatic invariants that are conserved in the vertically displaced portions of fluid, so that the density is also conserved. This allows us to determine density fluctuations, $\rho' = (\partial \rho / \partial z) \delta z$, and the fluctuations of potential energy per unit mass:

$$\delta E_P = \frac{g}{\rho_0} \int_z^{z+\delta z} \rho' \, \mathrm{d}z = \frac{1}{2} \frac{b'^2}{N^2}. \tag{19}$$

Let us consider the thermally stratified atmosphere, where the buoyancy, b, is expressed through the potential temperature, $b = \beta\theta$. Consequently the TPE is proportional to the energy of the potential temperature fluctuations:

$$E_p = \left(\frac{\beta}{N}\right)^2 E_\theta = \frac{1}{2}\left(\frac{\beta}{N}\right)^2 \langle\theta^2\rangle. \tag{20}$$

Then by multiplying Eq. (8b) by $(\beta/N)^2 = (\partial\Theta/\partial z)^{-1}$ and assuming that N changes only slowly compared to turbulent variations we arrive at the following TPE budget equation[1]:

$$\frac{\mathrm{D}E_P}{\mathrm{D}t} + \frac{\partial}{\partial z}\Phi_P = -\beta F_z - \varepsilon_P = -\beta F_z - \frac{E_P}{C_P t_T}, \tag{21}$$

where $\Phi_P = (\beta/N)^2\Phi_\theta$ and $\varepsilon_P = (\beta/N)^2\varepsilon_\theta$. The term βF_z appears in Eqs. (7b) and (21) with opposite signs and describes the energy exchange between TKE and TPE.

The sum of the TKE and TPE is simply the total turbulent energy (TTE):

$$E = E_K + E_P = \frac{1}{2}\left(\langle\mathbf{u}^2\rangle + \left(\frac{\beta}{N}\right)^2 \langle\theta^2\rangle\right), \tag{22}$$

and the TTE budget equation is immediately derived by summing up Eqs. (7b) and (21). Generally speaking, the time-scale constants C_K and C_P in Eq. (16), which characterise the kinetic and the potential energy dissipation rates, can differ. Here, for simplicity, we use $C_K = C_P$. Then the TTE budget equation becomes

$$\frac{DE}{Dt} + \frac{\partial}{\partial z}\Phi_T = -\tau_{i3}\frac{\partial U_i}{\partial z} - \frac{E}{C_K t_T}, \tag{23}$$

where $\Phi_T = \Phi_K + \Phi_P$ is the TTE vertical flux.

In the steady state, Eq. (23) reduces to a simple balance between the TTE production = τS (where $\tau^2 = \tau_{13}^2 + \tau_{23}^2$) and the TTE dissipation $\sim E t_T^{-1}$, which yields $E \sim \tau S t_T$. In Section 5 we demonstrate that for a very large Ri the ratios τ/E, E_K/E and E_z/E_K tend to become non-zero constants. In that case estimating t_T through the turbulent length scale, l_z, as $t_T \sim l_z E_z^{-1/2} \sim l_z E^{-1/2}$ yields an asymptotic large-Ri estimate, $E \sim (l_z S)^2 > 0$. This reasoning does not allow the existence of the critical Richardson number.

As a matter of interest, traditional analyses of the turbulent energy have been basically limited to using TKE budget, Eq. (7b). Equation (8b) for the squared potential temperature fluctuations, although it is well-known for decades, has been ignored in the operationally used turbulent closure models. Only rather recently, E_θ has been treated in terms of the TPE, see Dalaudier and Sidi (1987), Hunt et al. (1988), Canuto and Minotti (1993), Schumann and Gerz (1995), Hanazaki and Hunt (1996,2004), Keller and van Atta (2000), Stretch et al. (2001), Canuto et al. (2001), Cheng et al. (2002), Luyten et al. (2002, p. 257), Jin et al. (2003), Umlauf (2005) and Rehmann and Hwang (2005). Zilitinkevich (2002) employed the TKE and the TPE budget equations on equal terms to derive an energetically consistent turbulent closure model, avoiding the traditional hypothesis $K_H \sim K_M \sim E_K t_T$ (which leads to a dead end, at least in stable stratification). All three budgets, for TKE, TPE and TTE have been considered by Canuto and Minotti (1993) and Elperin et al. (2002).

[1] Alternatively the TPE budget equation can be derived from the equation for the fluctuation of buoyancy, b, namely, by multiplying this equation by bN^{-2}, and then applying statistical averaging. It follows then that Eq. (21) holds true independently of the assumption that N changes slowly.

4 Local model for the steady-state, homogeneous regime

4.1 Anisotropy of turbulence

In this section we consider the equilibrium turbulence regime and neglect the third-order transport terms, so that the left-hand sides (l.h.s.) in all budget equations become zero. We limit our analysis to boundary-layer type flows, in which the horizontal gradients of the mean velocity and temperature are negligibly small. For these conditions the TKE production rate becomes

$$\Pi = -\boldsymbol{\tau} \cdot \frac{\partial \mathbf{U}}{\partial z} = \tau S, \tag{24}$$

where $\tau = (\tau_{xz}, \tau_{yz}, 0)$, and $\tau \equiv |\boldsymbol{\tau}|$. It goes without saying that Π is determined differently in other types of turbulent flows, in particular in the wave boundary layer below the ocean surface or in the capping inversion layer above the long-lived atmospheric stable boundary layer, where the TKE is at least partially produced by the breaking of surface waves in water or internal gravity waves in the atmosphere. Note that in the laboratory conditions these mechanisms are similar to the oscillating-grid generation of turbulence rather than to turbulence generation by shear.

Taking $C_P = C_K$ [see discussion of Eq. (23) in Sect. 3], Eqs. (19)–(23) yield the following expressions for the turbulent energies:

$$E = C_K t_T \Pi, \tag{25a}$$

$$E_P = -C_K t_T \beta F_z = E\mathrm{Ri}_f, \tag{25b}$$

$$E_K = C_K t_T (\Pi + \beta F_z) \equiv C_K t_T \Pi (1 - \mathrm{Ri}_f) = E(1 - \mathrm{Ri}_f), \tag{25c}$$

where Ri_f is the familiar flux Richardson number defined as the ratio of the TKE consumption needed for overtaking buoyancy forces to the TKE production by the velocity shear:

$$\mathrm{Ri}_f \equiv -\frac{\beta F_z}{\Pi} = \frac{\mathrm{Ri}}{\mathrm{Pr}_T} = \frac{E_P}{E}. \tag{26}$$

The above analysis implies that Ri_f is then the ratio of TPE to TTE, a fact that has been overlooked until recently.[2] Ri_f is equal to zero in neutral stratification, monotonically increases with increasing stability, but obviously cannot exceed unity. Hence, for very strong static stability (at $\mathrm{Ri} \to \infty$) it must approach a non-zero, positive limit, $\mathrm{Ri}_f^\infty < 1$. This conclusion by no means supports the existence of the critical gradient Richardson number. Indeed, Ri_f is an internal parameter that is controlled by turbulence in contrast to $\mathrm{Ri} = (\beta\partial\Theta/\partial z)/(\partial \mathbf{U}/\partial z)^2$, which is an "external" parameter that characterises the mean flow.

It is worth recalling that the key parameter characterising vertical turbulent transports is the TKE of the vertical velocity fluctuations, $E_z = \frac{1}{2}\langle w^2\rangle$, rather than the full TKE. In order to determine E_z, we need to consider all three budget equations (10a) for the diagonal Reynolds stresses, $\tau_{11} = 2E_1 = 2E_x = \langle u^2\rangle$, $\tau_{22} = 2E_2 = 2E_y = \langle v^2\rangle$ and $\tau_{33} = 2E_3 = 2E_z = \langle w^2\rangle$. In the steady state these budget equations become

$$\frac{E_i}{C_K t_T} = -\tau_{i3}\frac{\partial U_i}{\partial z} + \frac{1}{2}Q_{ii}, \quad i = 1, 2, \tag{27a}$$

$$\frac{E_z}{C_K t_T} = \frac{E_3}{C_K t_T} = \beta F_z + \frac{1}{2}Q_{33}. \tag{27b}$$

[2] Taking into account that C_P and C_K can differ, Ri_f is proportional rather than equal to E_P/E.

The sum of the pressure-velocity shear correlation terms, $\sum Q_{ii} = \sum \rho_0^{-1} \langle p\partial u_i/\partial x_i \rangle$, is zero because of the continuity equation, $\sum \partial u_i/\partial x_i = 0$. Hence, they are neither productive nor dissipative; they simply describe the conversion of the energy of "rich" components into the energy of "poorer" components.

In order to determine Q_{11}, Q_{22} and Q_{33}, we generalize the familiar "return-to-isotropy" hypothesis as follows:

$$Q_{11} = -\frac{2C_r}{3C_K t_T}(3E_1 - E_K \Psi_1), \tag{28a}$$

$$Q_{22} = -\frac{2C_r}{3C_K t_T}(3E_2 - E_K \Psi_2), \tag{28b}$$

$$Q_{33} = -\frac{2C_r}{3C_K t_T}(3E_3 - E_K \Psi_3). \tag{28c}$$

Here, C_r and Ψ_i (i = 1, 2, 3) are dimensionless empirical coefficients; C_r accounts for the difference between the relaxation-time and the dissipation-time scales (as a first approximation, we take these two time scales to be proportional, $t_r \sim t_T$, so that $C_r = t_r/t_T =$ constant); Ψ_i govern redistribution of TKE between the components. When $\Psi_i = 1$ the above relations reduce to their original form (Rotta 1951) and are known to be a good approximation for neutrally stratified flows. In stable stratification, we need to leave room for their possible stability dependence. As a first approximation, we assume

$$\Psi_i = 1 + C_i \mathrm{Ri}_f, \quad i = 1, 2, 3, \tag{29}$$

where Ri_f is the flux Richardson number, and C_i are empirical constants. Their sum must be zero, $C_1 + C_2 + C_3 = 0$, in order to satisfy the condition $\sum Q_{ii} = 0$ (which is needed to guarantee that $E_K = E_1 + E_2 + E_3$). Linear functions of Ri_f on the r.h.s. of Eq. (29) are taken as simple approximations providing the only possible (from the physical point of view) finite, non-zero limits: Ψ_i = 1 at Ri = 0, and $\Psi_i \to 1 + C_i \mathrm{Ri}_f^\infty$ at $\mathrm{Ri} \to \infty$.

Because the energy exchange between the horizontal components of TKE, E_1 and E_2, is not directly affected by the stable stratification, we take the first two energy-exchange constants to be equal, $C_1 = C_2$. Then, the condition $C_1 + C_2 + C_3 = 0$ implies that only one of the three constants is independent and $C_1 = C_2 = -\frac{1}{2}C_3$.

Equations (27)–(28) yield

$$E_i = \frac{C_r}{3(1+C_r)} E_K \Psi_i - \frac{C_K}{1+C_r} t_T \tau_{i3} \frac{\partial U_i}{\partial z}, \quad i = 1, 2, \tag{30a}$$

$$E_z = \frac{C_r}{3(1+C_r)} E_K \Psi_3 + \frac{C_K}{1+C_r} t_T \beta F_z. \tag{30b}$$

In the plain-parallel neutral boundary layer with $\mathbf{U} = (U, 0, 0)$, Eqs. (30a) and (30b) reduce to

$$\frac{E_x}{E_K} = \frac{3 + C_r}{3(1+C_r)}, \tag{31a}$$

$$\frac{E_y}{E_K} = \frac{E_z}{E_K} = \frac{C_r}{3(1+C_r)}. \tag{31b}$$

Given the vertical component of TKE, E_z, the turbulent dissipation time scale, $t_T = l_T E_K^{-1/2}$, can alternatively be expressed through the vertical turbulent length scale l_z:

$$t_T = \frac{l_z}{E_z^{1/2}}. \tag{32}$$

Then eliminating t_T from Eq. (25c) and Eq. (30b), and substituting Eq. (29) for Ψ_3 yields

$$E_z = \left[\frac{C_K C_r \Psi_3}{3(1+C_r)}\left(\Pi + \left(\frac{3}{C_r \Psi_3}+1\right)\beta F_z\right) l_z\right]^{2/3}, \tag{33a}$$

$$\Psi_3 = 1 + C_3 \mathrm{Ri}_f. \tag{33b}$$

This formulation recovers the traditional return-to-isotropy formulation when $C_3 = 0$.

In order to close the system, the horizontal components of the TKE, E_x and E_y, are not required. We leave the discussion of these components to a separate paper, in which our closure is extended to passive scalars and applied to turbulent diffusion.

4.2 Vertical turbulent fluxes of momentum and potential temperature

Of the non-diagonal Reynolds stresses we consider only those representing the vertical fluxes of momentum $\tau_{13} = \tau_{xz} = \langle uw \rangle$ and $\tau_{23} = \tau_{yz} = \langle vw \rangle$ which are needed to close the momentum equations (4)–(5) and are determined by Eq. (10b). In the steady state, using Eqs. (17)–(18) for the effective Reynolds-stress dissipation rate, we obtain the following relation for the non-diagonal Reynolds stresses:

$$\tau_{i3} = -2\Psi_\tau E_z^{1/2} l_z \frac{\partial U_i}{\partial z}. \tag{34}$$

Likewise, of the three components of the potential-temperature flux, we consider only the vertical flux $F_3 = F_z$ that is needed to close the thermodynamic energy Equation (6). The vertical flux F_z is determined by Eq. (9b). Taking $\beta E_\theta = (N^2/\beta) E_P = -C_K N^2 l_z F_z / E_z^{1/2}$ [after Eqs. (25b) and (32)], the steady-state version of Eq. (9b) becomes

$$F_z = -\frac{2 C_F E_z^{1/2} l_z}{1 + 2 C_\theta C_F C_K (N l_z)^2 E_z^{-1}} \left(\frac{\partial \Theta}{\partial z}\right). \tag{35}$$

Substituting here $N^2 = \beta \partial\Theta/\partial z$ shows that F_z depends on $\partial\Theta/\partial z$ weaker than linearly and at $\partial\Theta/\partial z \to \infty$ tends to a finite limit:

$$F_{z,\max} = -\frac{E_z^{3/2}}{C_\theta C_K \beta l_z}. \tag{36}$$

It follows then that F_z in a turbulent flow cannot be considered as a given external parameter. This conclusion is consistent with our reasoning in Sect. 4.1 that the flux Richardson number $\mathrm{Ri}_f = -\beta F_z (\tau S)^{-1}$ is an internal parameter of turbulence that cannot be arbitrarily prescribed. According to Eq. (36), the maximum value of the buoyancy flux βF_z, in the strong stability limit, is proportional to the dissipation rate, $E_z^{3/2} l_z^{-1}$, of the energy of vertical velocity fluctuations.[3]

Equations (34) and (35) allow us to determine the eddy viscosity and conductivity:

$$K_M \equiv \frac{-\tau_{i3}}{\partial U_i/\partial z} = 2\Psi_\tau E_z^{1/2} l_z, \tag{37a}$$

$$K_H \equiv \frac{-F_z}{\partial\Theta/\partial z} = \frac{2 C_F E_z^{1/2} l_z}{1 + 2 C_\theta C_F C_K (N l_z)^2 E_z^{-1}}. \tag{37b}$$

[3] A principally similar analysis of the budget equation for F_z has been performed by Cheng et al. (2002). Their Eq. (15i) implies the same maximum value of F_z as our Eq. (36). It worth noting that Eq. (35) imposes an upper limit on the downward heat flux in the deep ocean (which is known to be a controlling factor of the rate of the global warming).

Consequently, the Kolmogorov closure hypothesis applied to the effective Reynolds-stress dissipation rate, Eqs. (17)–(18), yields the eddy-viscosity formulation, Eq. (37a), basically similar to the traditional formulation, Eq. (1), whereas Eq. (37b) for eddy conductivity differs essentially from this formulation.

It may appear that our derivation of Eq. (37a) essentially depends on the hypothetical concept of the effective dissipation rate, Eqs. (17)–(18). Actually we employ this merely for the reader's convenience, to avoid overly complex derivations. Principally, the same result, namely the downgradient momentum-flux formulation equivalent to Eqs. (34) and (37a), follows from analyses of the budget equations for the Reynolds stresses in the k-space using the familiar "τ-approximation" (see, e.g., Elperin et al. 2002, 2006).

Recall now that Ψ_τ is a dimensionless, non-zero, bounded coefficient that can only monotonically depend on the static stability [see Eqs. (17)–(18) and their discussion in Sect. 2]. Let us approximate the stability dependence of Ψ_τ by the following linear function of the flux Richardson number, Ri_f:

$$\Psi_\tau = C_{\tau 1} + C_{\tau 2}\mathrm{Ri}_f, \tag{38}$$

where $C_{\tau 1}$ and $C_{\tau 2}$ are dimensionless constants to be determined empirically. Equation (38) provides the only physically meaningful, finite, non-zero limits, namely, $\Psi_\tau = C_{\tau 1}$ at $\mathrm{Ri} = 0$, and $\Psi_\tau \to C_{\tau 1} + C_{\tau 2}\mathrm{Ri}_f^\infty$ at $\mathrm{Ri} \to \infty$ [cf. our argument in support of Eq. (29)].

4.3 Turbulent Prandtl number and other dimensionless parameters

The system of Eq. (33a)–(35), although unclosed until we determine the vertical turbulent length scale l_z, reveals a "partial invariance" with respect to l_z and allows determining the turbulent Prandtl number, Pr_T, the flux Richardson number, Ri_f, and other dimensionless characteristics of turbulence in the form of universal functions of the gradient Richardson number, Ri. Obviously such universality is relevant only to the steady-state homogeneous regime. In non-steady, heterogeneous regimes, all these characteristics are not single-valued functions of Ri.

Recalling that $\Pi = K_M S^2$ and $\mathrm{Ri}_f \equiv -\beta F_z/\Pi$, Eqs. (33a) and (37a) give

$$\frac{E_z}{(Sl_z)^2} = \Psi(\mathrm{Ri}_f) \equiv \frac{2C_K C_r \Psi_3 \Psi_\tau}{3(1+C_r)}\left[1 - \left(\frac{3}{C_r\Psi_3} + 1\right)\mathrm{Ri}_f\right], \tag{39}$$

where Ψ_3 and Ψ_τ are linear functions of Ri_f given by Eqs. (33b) and (38). Then dividing K_M [determined by Eq. (37a)] by K_H [determined by Eq. (37b)] and expressing E_z through Eq. (39) yields the following surprisingly simple expressions:

$$\mathrm{Pr}_T \equiv \frac{K_M}{K_H} = \frac{\mathrm{Ri}}{\mathrm{Ri}_f} = \frac{\Psi_\tau}{C_F} + \frac{3(1+C_r)C_\theta}{C_r\Psi_3}\mathrm{Ri}\left[1 - \left(\frac{3}{C_r\Psi_3} + 1\right)\mathrm{Ri}_f\right]^{-1}, \tag{40}$$

and

$$\frac{1}{\mathrm{Ri}} = \frac{C_F\Psi_\tau^{-1}}{\mathrm{Ri}_f} - \frac{3C_F(1+C_r)C_\theta\Psi_\tau^{-1}}{C_r\Psi_3(1-\mathrm{Ri}_f) - 3\mathrm{Ri}_f}, \tag{41}$$

which do not include l_z. Equation (41) together with Eqs. (33b) and (38) specify Ri as a single-valued, monotonically increasing function of Ri_f determined in the interval $0 < \mathrm{Ri}_f < \mathrm{Ri}_f^\infty$, where Ri_f^∞ is given by Eq. (45). Therefore, the inverse function, namely,

$$\mathrm{Ri}_f = \Phi(\mathrm{Ri}), \tag{42}$$

is a monotonically increasing function of Ri, changing from 0 at $\mathrm{Ri} = 0$ to Ri_f^∞ at $R_i \to \infty$.

According to the above equations, the Ri dependencies of Ri_f and Pr_T (which is also a monotonically increasing function of Ri) are characterised by the following asymptotic limits:

$$\mathrm{Pr}_T \approx \frac{\Psi_\tau^{(0)}}{C_F} + \left(\frac{3C_\theta(1+C_r)}{C_r} + \frac{C_{\tau 2}}{C_F}\right)\mathrm{Ri} \quad \rightarrow \mathrm{Pr}_T^{(0)} = \frac{\Psi_\tau^{(0)}}{C_F}, \tag{43a}$$

$$\mathrm{Ri}_f \approx \frac{C_F}{\Psi_\tau^{(0)}}\mathrm{Ri} \quad \text{at} \quad Ri \ll 1, \tag{43b}$$

$$\mathrm{Pr}_T \approx \frac{1}{\mathrm{Ri}_f^\infty}\mathrm{Ri}, \tag{44a}$$

$$\mathrm{Ri}_f \rightarrow \mathrm{Ri}_f^\infty \quad \text{at} \quad \mathrm{Ri} \gg 1, \tag{44b}$$

where

$$\mathrm{Ri}_f^\infty = \frac{C_r\Psi_3^\infty}{C_r\Psi_3^\infty + 3[1 + C_\theta(1+C_r)]}, \tag{45}$$

and the superscripts "(0)" and "∞" mean "at Ri = 0" and "at Ri$\rightarrow \infty$", respectively.

Equations (33a)–(35) allow us to determine, besides Pr_T, three other dimensionless parameters the vertical anisotropy of turbulence:

$$A_z \equiv \frac{E_z}{E_K} = \frac{C_r\Psi_3}{3(1+C_r)}\left[1 - \left(\frac{3}{C_r\Psi_3} + 1\right)\mathrm{Ri}_f\right](1 - \mathrm{Ri}_f)^{-1}, \tag{46}$$

the squared ratio of the turbulent flux of momentum to the TKE (which characterises the correlation between vertical and horizontal velocity fluctuations):

$$\left(\frac{\tau}{E_K}\right)^2 = \frac{2\Psi_\tau A_z}{C_K(1 - \mathrm{Ri}_f)}, \tag{47}$$

and the ratio of the squared vertical flux of potential temperature to the product of the TKE and the "energy" of the potential temperature fluctuations:

$$\frac{F_z^2}{E_K E_\theta} = \frac{2\Psi_\tau A_z}{C_K\,\mathrm{Pr}_T}. \tag{48}$$

Equations (41), (46)–(48) determine the Ri dependencies of A_z, $\tau^2 E_K^{-2}$ and $F_z^2(E_K E_\theta)^{-2}$, which are characterised by the following asymptotic limits:

$$A_z \rightarrow A_z^{(0)} = \frac{C_r}{3(1+C_r)}, \tag{49a}$$

$$\left(\frac{\tau}{E_K}\right)^2 \rightarrow \frac{2\Psi_\tau^{(0)}A_z^{(0)}}{C_K}, \tag{49b}$$

$$\frac{F_z^2}{E_K E_\theta} \rightarrow \frac{2C_F A_z^{(0)}}{C_K} \quad \text{at} \quad \mathrm{Ri} \ll 1, \tag{49c}$$

$$A_z \to A_z^\infty = C_\theta Ri_f^\infty (1 - Ri_f^\infty)^{-1}, \tag{50a}$$

$$\left(\frac{\tau}{E_K}\right)^2 \to \frac{2\Psi_\tau^\infty A_z^\infty}{C_K(1 - \mathrm{Ri}_f^\infty)}, \tag{50b}$$

$$\frac{F_z^2}{E_K E_\theta} \to \frac{2\Psi_\tau^\infty A_z^\infty}{C_K \, \mathrm{Pr}_T^\infty} \quad \text{at} \quad \mathrm{Ri} \gg 1. \tag{50c}$$

It must be noted that the turbulent velocity scale in Eqs. (34)–(37a) is $\sqrt{E_z}$ rather than $\sqrt{E_K}$. However, in a number of currently used turbulence closure models the stability dependence of $A_z = E_z/E_K$ is neglected and $\sqrt{E_K}$ is taken as an ultimate velocity scale to characterise the vertical turbulent transports. This is done unfortunately without serious theoretical or experimental grounds. On the contrary, Eq. (46) implies an essential Ri dependence of A_z, which is in agreement with currently available data [see Mauritsen and Svensson (2007) and our data analysis in Sect. 5 below].

4.4 Vertical turbulent length scale

Two basic factors impose limits on the vertical turbulent length scale, l_z, in geophysical flows: the height over the surface (the geometric limit) and the stable stratification.

In neutral stratification, l_z is restricted by the geometric limit[4]:

$$l_z \sim z. \tag{51}$$

For the strong stable stratification limit, different formulations have been proposed. Monin and Obukhov (1954) proposed the following length scale widely used in boundary-layer meteorology:

$$L \equiv \frac{\tau^{3/2}}{-\beta F_z} = \frac{\tau^{1/2}}{S\mathrm{Ri}_f}. \tag{52}$$

Our local closure model is consistent with this limit: Eqs. (34), (39) and (52) yield

$$l_z = \frac{\mathrm{Ri}_f}{(2\Psi_\tau)^{1/2}\Psi^{1/4}} L. \tag{53}$$

Furthermore any interpolation formula for l_z linking the limits $l_z \sim z$ and $l_z \sim L$ should have the form

$$l_z = z\Psi_l(\mathrm{Ri}_f), \tag{54}$$

where Ψ_l is a function of Ri_f.

Well-known alternatives to L are the Ozmidov scale: $\varepsilon_K^{1/2} N^{3/2}$ (Ozmidov 1990); the local energy balance scale: $E_z^{1/2} N^{-1}$ (e.g., Table 3 in Cuxart et al. 2006); and the shear sheltering scale: $E_z^{1/2} S^{-1}$ (Hunt et al. 1985, 1988). Using our local closure equations (Sect. 4.1–4.3) the ratio of each of these scales to L can be expressed through a corresponding function of Ri_f. Hence, any interpolation linking the neutral stratification limit, $l_z \sim z$, with all the above limits will still have the same form as Eq. (54).

---

[4] In rotating fluids, the direct effect of the angular velocity, Ω, on turbulent eddies is characterised by the rotational limit, $E_z^{1/2}/\Omega$. In geophysical, stably stratified flows it has only a secondary importance. We leave the discussion of this effect for future work.

Consequently, Eq. (54) represents a general formulation for the vertical turbulent length scale in the steady-state, homogeneous, stably stratified flows. In other words, the stability dependence of l_z is fully characterised by the universal function $\Psi_l(\mathrm{Ri}_f)$. This function should satisfy the following physical requirements: in neutral stratification it attains the maximum value, $\Psi_l(0) = 1$ [the omitted empirical constant combines with the coefficients $C_K = C_P$, C_F and Ψ_τ in Eqs. (16) and (18)], and with increasing Ri_f it should monotonically decrease. Finally, at $\mathrm{Ri}_f \to \mathrm{Ri}_f^\infty$ this function should tend to zero [otherwise Eq. (33a) would give $E_z > 0$ at $\mathrm{Ri}_f \to \mathrm{Ri}_f^\infty$, which is physically senseless].

We propose a simple approximation to satisfy these requirements: $\Psi_l = \left(1 - \mathrm{Ri}_f/\mathrm{Ri}_f^\infty\right)^n$, where n is a positive constant. Using an empirical value of n = 4/3 (see the next Section) we arrive at the following relation for l_z:

$$l_z = z\left(1 - \frac{\mathrm{Ri}_f}{\mathrm{Ri}_f^\infty}\right)^{4/3}. \tag{55}$$

Obviously, in non-steady, heterogeneous regimes l_z should be determined through a prognostic equation accounting for its advection and temporal evolution.

5 Comparison of the local model with experimental and numerically simulated data

To determine the empirical dimensionless constants $C_r, C_K, C_F, C_\theta, C_{\tau 1}, C_{\tau 2}, C_3$ and n we compare results from the local closure model presented in Sect. 4 with experimental, large-eddy simulation (LES) and direct numerical simulation (DNS) data.

As mentioned earlier, the local model is applied to homogeneous turbulence and does not include transports of turbulent energies and turbulent fluxes. At the same time practically all currently available data represent vertically (in a number of cases, both vertically and horizontally) heterogeneous flows, in which the above transports are more or less pronounced. In these conditions, fundamental dimensionless parameters of turbulence, such as Pr_T, Ri_f, $(\tau/E_K)^2$, $F_z^2/(E_K E_\theta)$ and A_z, can be only approximately considered as universal functions of Ri. Mauritsen and Svensson (2007) have demonstrated quite reasonable Ri dependencies of the above parameters based on datasets obtained in several recent field campaigns. To reduce inevitable deviations from universality and to more accurately determine empirical constants, we now more carefully select data and rule out those that represent strongly heterogeneous regimes.

Figures 1a, b show the turbulent Prandtl number, Pr_T, and flux Richardson number, $\mathrm{Ri}_f = \mathrm{Ri}/\mathrm{Pr}_T$, versus the gradient Richardson number, Ri. They demonstrate reasonable agreement between data from atmospheric and laboratory experiments, LES and DNS. Data for $\mathrm{Ri} \to 0$ in Fig. 1 are consistent with the commonly accepted empirical estimate of $\mathrm{Pr}_T^{(0)} \equiv \mathrm{Pr}_T|_{\mathrm{Ri}\to 0} = 0.8$ [see data collected by Churchill (2002) and Foken (2006) and the theoretical analysis of Elperin et al. (1996)]. Figure 1b clearly demonstrates that Ri_f at large Ri levels off, allowing an estimate of its limiting value, $\mathrm{Ri}_f^\infty = 0.2$.

Figure 2 shows Ri dependencies of the dimensionless turbulent fluxes: (a) $\hat{\tau}^2 \equiv (\tau/E_K)^2$ and (b) $\hat{F}_z^2 \equiv F_z^2/(E_K E_\theta)$. It is long recognised [see, e.g., Sect. 5.3 and 8.5 in Monin and Yaglom (1971)] that in neutral stratification, atmospheric observations give more variable and generally smaller values of these dimensionless turbulent fluxes than laboratory experiments. This is not surprising because measured values of the TKE, E_K, in the atmosphere are contaminated with low-frequency velocity fluctuations caused by the interaction of the airflow

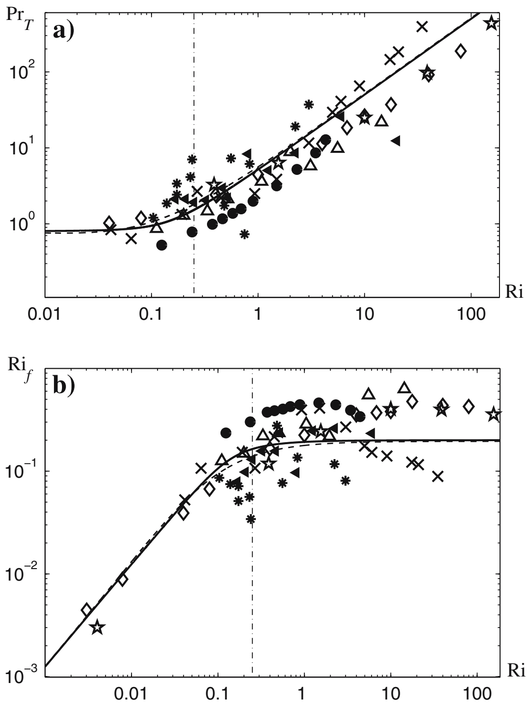

Fig. 1 Ri dependences of (**a**) turbulent Prandtl number, $\mathrm{Pr}_T = K_M/K_H$, and (**b**) flux Richardson number, $\mathrm{Ri}_f = -\beta F_z(\tau S)^{-1}$, based on meteorological observations: slanting black triangles (Kondo et al. 1978), snowflakes (Bertin et al. 1997); laboratory experiments: black circles (Strang and Fernando 2001), slanting crosses (Rehmann and Koseff 2004), diamonds (Ohya 2001); LES: triangles (new data provided by Igor Esau); DNS: five-pointed stars (Stretch et al. 2001). Solid lines show our model for homogeneous turbulence; dashed line, analytical approximations after Eq. (64)

with surface heterogeneities. These low-frequency fluctuations, however, should not be confused with shear-generated turbulence. Therefore, to validate our turbulence closure model it is only natural to use data on $\hat{\tau}^2$ obtained from laboratory experiments and/or numerical simulations. Relying on these data presented in Fig. 2a, we obtain $(\tau/E_K)^{(0)} = 0.326$ for Ri $\ll$ 1; and $(\tau/E_K)^{\infty}$= 0.18 for Ri $\gg$ 1 [the superscripts "(0)" and "∞" mean "at Ri = 0" and "at Ri→ ∞"]. These estimates are consistent with the conditions $(\hat{\tau}^2)^{(0)}/(\hat{F}_z^2)^{(0)} = \mathrm{Pr}_T^{(0)} = 0.8$, and $(\hat{F}_z^2)^{\infty} = 0$ that follow from Eqs. (47)–(48). Furthermore, Fig. 3, which shows the Ri dependencies of the re-normalised fluxes, (a) $\hat{\tau}^2/(\hat{\tau}^2)^{(0)}$ and (b) $\hat{F}_z^2/(\hat{F}_z^2)^{(0)}$, reveals essential similarity in the shape of these dependencies based on atmospheric, laboratory and LES data, and provides additional support to our analysis.

Data on the vertical anisotropy of turbulence, $A_z = E_z/E_K$, are shown in Fig. 4. These data are quite ambiguous and need to be analysed carefully. For neutral stratification, we adopt the estimate of $A_z^{(0)} = 0.25$ based on precise results from laboratory experiments (Agrawal et al. 2004) and DNS (Moser et al. 1999). These data are now commonly accepted and have been shown to be consistent with independent data on the wall-layer turbulence (L'vov et al. 2006). Current and previous atmospheric data (e.g., those shown in Fig. 75 in Monin and Yaglom 1971) yield smaller values of $A_z^{(0)}$, but, as already mentioned, they overestimate the horizontal TKE and, consequently, underestimate A_z, especially in neutral stratification. Such overestimating arises from meandering of atmospheric boundary-layer flow caused by non-uniform features of the earth's surface (hills, houses, groups of trees, etc.). At the same time, very large values of Ri in currently available experiments and numerical simulations are relevant to turbulent flows above the boundary layer, where the TKE of local origin (controlled by the local Ri) is often small compared to the TKE transported from

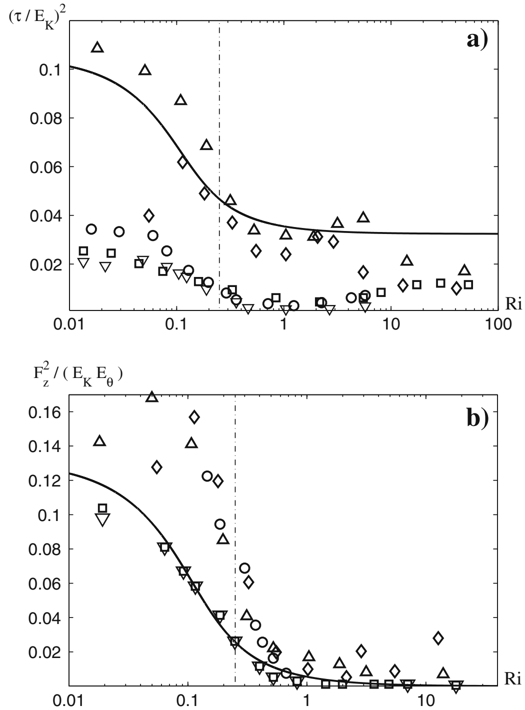

Fig. 2 Same as in Fig. 1 but for the squared dimensionless turbulent fluxes of (**a**) momentum, $\hat{\tau}^2 = (\tau/E_K)^2$, and (b) potential temperature, $\hat{F}_z^2 = F_z^2/(E_K E_\theta)$, based on laboratory experiments: diamonds (Ohya 2001) and LES: triangles (new data provided by Igor Esau); and meteorological observations: squares [CME = Carbon in the Mountains Experiment, Mahrt and Vickers (2005)], circles [SHEBA = Surface Heat Budget of the Arctic Ocean, Uttal et al. (2002)] and overturned triangles [CASES-99 = Cooperative Atmosphere-Surface Exchange Study, Poulos et al. (2002), Banta et al. (2002)]

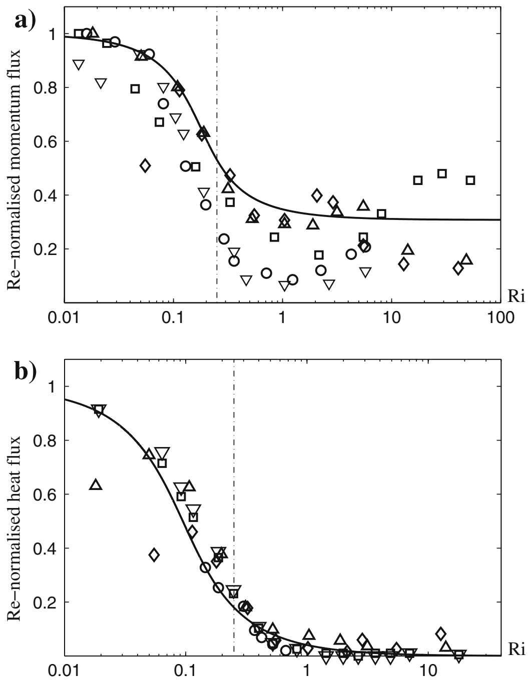

Fig. 3 Same as in Fig. 2 but for re-normalised turbulent fluxes: (**a**) $\hat{\tau}^2/(\hat{\tau}^2)^{(0)}$ and (**b**) $\hat{F}_z^2/(\hat{F}_z^2)^{(0)}$, where the superscript (0) indicates mean values at Ri = 0 [hence $\hat{\tau}^2/(\hat{\tau}^2)^{(0)}$ and $\hat{F}_z^2/(\hat{F}_z^2)^{(0)}$ equal 1 at Ri = 0]

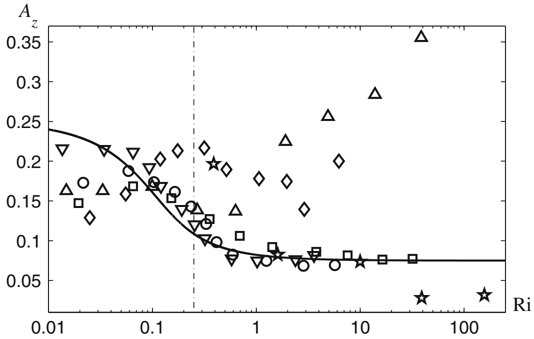

Fig. 4 Same as in Figure 2 but for the vertical anisotropy of turbulence, $A_z = E_z/E_K$, on addition of DNS data of Stretch et al. (2001) shown by five-pointed stars

the lower, strong-shear layers. It is not surprising that the spread of data on A_z versus Ri is quite large. Nonetheless, atmospheric data characterise A_z as a monotonically decreasing function of Ri and allow at least an approximate estimate of its lower limit: $A_z^\infty = 0.075$.

Below we use the estimates of $A_z^{(0)}$, $(\tau^2 E_K^{-2})_{\mathrm{Ri}=0}$, $\mathrm{Pr}_T^{(0)}$, Ri_f^∞, A_z^∞, and $(\tau^2 E_K^{-2})_{\mathrm{Ri}=\infty}$ to determine our empirical constants.

We start with data for neutral stratification. The empirical estimate of $A_z^{(0)} = 0.25$ yields

$$C_r = 3A_z^{(0)}(1 - 3A_z^{(0)})^{-1} = 3. \tag{56}$$

Then we combine Eq. (25c) for E_K with Eq. (32) for t_T and consider the logarithmic boundary layer, in which $l_z = z, \tau = \tau|_{z=0} \equiv u_*^2$ and $S = u_*(kz)^{-1}$(u_* is the friction velocity and k is the von Karman constant) to obtain

$$C_K = k(A_z^{(0)})^{1/2} \left(\frac{E_K}{\tau}\right)^{3/2}_{\mathrm{Ri}=0} = 1.08. \tag{57}$$

This estimate is based on the well-determined empirical value of $k \approx 0.4$, and the above values of $(\tau/E_K)^{(0)} = 0.326$ and $A_z^{(0)} = 0.25$. Then taking $C_K = 1.08$, $\mathrm{Pr}_T^{(0)} = 0.8$ and using Eqs. (43a) and (47) we obtain

$$C_{\tau 1} = \frac{C_K}{2A_z^{(0)}} \left(\frac{E_K}{\tau}\right)^{-2}_{\mathrm{Ri}=0} = 0.228, \tag{58}$$

$$C_F = C_{\tau 1}/\mathrm{Pr}_T^{(0)} = 0.285. \tag{59}$$

Taking $C_r = 3$, $A_z^\infty = 0.075$, $\mathrm{Ri}_f^\infty = 0.2$ and using Eq. (46) we obtain

$$\Psi_3^\infty = \frac{A_z^\infty}{A_z^{(0)}} + \frac{3\mathrm{Ri}_f^\infty}{C_r(1 - \mathrm{Ri}_f^\infty)} = 0.55; \quad C_3 = \frac{1}{\mathrm{Ri}_f^\infty}\left(\Psi_3^\infty - 1\right) = -2.25. \tag{60}$$

The constants C_1 and C_2 determine only the energy exchange between the horizontal velocity components and do not affect any other aspects of our closure model. Taking them equal (based on symmetry reasons) and recalling that $C_1 + C_2 + C_3 = 0$ yields

$$C_1 = C_2 = -\frac{1}{2}C_3 = 1.125. \tag{61}$$

Taking C_K= 1.08, Ri_f^∞= 0.2, A_z^∞= 0.075 and $(\tau/E_K)^\infty$= 0.18, Eq. (50b) gives

$$\Psi_\tau^\infty = \frac{C_K\left[(\tau/E_K)^\infty\right]^2(1-\mathrm{Ri}_f^\infty)}{2A_z^\infty} = 0.187; \quad C_{\tau 2} = \frac{1}{\mathrm{Ri}_f^\infty}\left(\Psi_\tau^\infty - C_{\tau 1}\right) = -0.208 \quad (62)$$

Then C_θ is determined from Eq. (45) written in the strong stability limit:

$$C_\theta = \frac{1}{1+C_r}\left[\frac{C_r\Psi_3^\infty}{3}\left(\frac{1}{\mathrm{Ri}_f^\infty} - 1\right) - 1\right] = 0.3. \quad (63)$$

Using the above values of dimensionless constants, the function $\mathrm{Ri}_f = \Phi(\mathrm{Ri})$ determined by Eqs. (41)–(42) is shown in Fig. 1b by the solid line. For practical use we propose the following explicit approximation of this function (with 5% accuracy):

$$\mathrm{Ri}_f = \Phi(\mathrm{Ri}) \approx 1.25\mathrm{Ri}\frac{(1\ +\ 36\mathrm{Ri})^{1.7}}{(1\ +\ 19\mathrm{Ri})^{2.7}}, \quad (64)$$

which is shown in Fig. 1b by the dashed line.

In the above estimates we did not use data on the dimensionless heat flux $\hat{F}_z^2 \equiv F_z^2/(E_K E_\theta)$ shown in Fig. 2b and 3b. The good correspondence between data and the theoretical curves in these figures serves as an empirical confirmation to our model.

The last empirical constant to be determined is the exponent n in Eq. (55). We eliminate l_z from Eqs. (53) and (54) to obtain

$$\frac{z}{L} = \frac{\mathrm{Ri}_f}{(2\Psi_\tau)^{1/2}\Psi^{1/4}\Psi_l}, \quad (65)$$

where Ψ_τ and Ψ are functions of Ri_f as specified by Eqs. (38) and (39). Given the dependence $\Psi_l(\mathrm{Ri}_f)$, the Eqs. (41) and (65) allow us to determine Ri_f and Ri as single-valued functions of z/L. V*ice versa*, given, e.g., the dependence $\mathrm{Ri}(z/L)$, Ψ_l can be determined as a single-valued function of Ri_f.

We can apply this analysis to deduce $\Psi_l(\mathrm{Ri}_f)$ from the empirical dependence of Ri on z/L obtained by Zilitinkevich and Esau (2007) using LES DATABASE64 (Esau 2004; Beare et al. 2006; Esau and Zilitinkevich 2006) and data from the field campaign SHEBA (Uttal et al. 2002). In Fig. 5, we present the above LES data together with our approximation based on Eq. (55). The exponent $n = 4/3$ is obtained from the best fit of the theoretical curve to all these data.

Strictly speaking, the suggested local, algebraic closure model is applicable only to homogeneous flows, in particular, to the nocturnal stable atmospheric boundary layer (ABL) of depth h, where non-local vertical turbulent transport plays a comparatively minor role, whereas $\boldsymbol{\tau}$ and F_z are reasonably accurately represented by universal functions of z/h (see, e.g., Fig. 1 in Zilitinkevich and Esau 2005); or with more confidence to the lower 10% of the ABL, the so-called surface layer, where τ and F_z can be taken depth-constant: $\tau \approx \tau|_{z=0} = u_*^2$ and $F_z \approx F_z|_{z=0} = F_*$.

As mentioned earlier, Eqs. (64) and (65) determine Ri_f and Ri as single-valued functions of z/L:

$$\mathrm{Ri}_f = \Phi_{\mathrm{Rif}}\left(\frac{z}{L}\right), \quad (66a)$$

$$\mathrm{Ri} = \Phi_{\mathrm{Ri}}\left(\frac{z}{L}\right). \quad (66b)$$

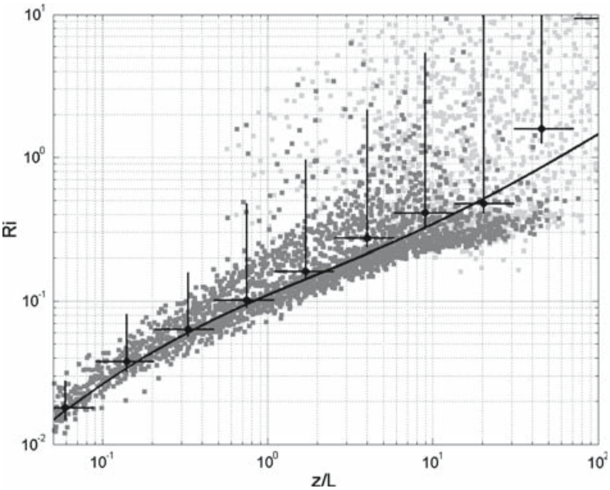

Fig. 5 Gradient Richardson number, Ri $=\beta(\partial\Theta/\partial z)(\partial U/\partial z)^{-2}$, versus dimensionless height z/L in the nocturnal atmospheric boundary layer (ABL). Dark- and light-grey points show LES data within and above the ABL, respectively; heavy black points with error bars are bin-averaged values of Ri [from Fig. 3 of Zilitinkevich and Esau (2007)]. Solid line is calculated after Eqs. (41), (55) and (65) with n= 4/3

Consequently, our model applied to the steady-state, homogeneous regime in the surface layer, is consistent with the similarity theory of Monin and Obukhov (1954). Given τ and F_z, this model allows us to determine z/L dependencies of all the dimensionless parameters considered above, as well as the familiar similarity-theory functions specifying mean velocity and temperature profiles:

$$\Phi_M \equiv \frac{kz}{\tau^{1/2}}\left(\frac{\partial U}{\partial z}\right) \equiv \frac{k}{\mathrm{Ri}_f}\left(\frac{z}{L}\right) = \frac{k}{\Phi_{\mathrm{Rif}}(z/L)}\left(\frac{z}{L}\right), \tag{67a}$$

$$\Phi_H \equiv \frac{k_T z\tau^{1/2}}{-F_z}\left(\frac{\partial\Theta}{\partial z}\right) \equiv k_T\frac{\mathrm{Pr}_T}{\mathrm{Ri}_f}\left(\frac{z}{L}\right) \equiv k_T\frac{\mathrm{Ri}}{\mathrm{Ri}_f^2}\left(\frac{z}{L}\right)$$

$$= k_T\frac{\Phi_{\mathrm{Ri}}(z/L)}{\Phi^2_{\mathrm{Rif}}(z/L)}\left(\frac{z}{L}\right), \tag{67b}$$

where k is the von Karman constant expressed through our constants by Eq. (57) and $k_T = k/\mathrm{Pr}_T^{(0)}$. At Ri $\ll 1$, Eqs. (66a, b) reduce to $\mathrm{Ri}_f \approx kz/L$ and $\mathrm{Ri} \approx \mathrm{Pr}_T^{(0)}\, kz/L$ while Eqs. (67a, b) recover the familiar wall-layer formulation. $\Phi_M(z/L)$ and $\Phi_H(z/L)$ calculated according to our model are shown in Figure 6 together with LES data from Zilitinkevich and Esau (2007).

In contrast to the commonly accepted paradigm that both Φ_M and Φ_H depend on z/L linearly, LES data and our solution show different asymptotic behaviours, namely, linear for Φ_M and stronger than linear for Φ_H. This result deserves discussion. Indeed, the traditional formulation, $\Phi_M, \Phi_H \sim z/L$ at $z/L \gg 1$, implies that Pr_T levels off (rather than increases) with increase of z/L and, as a consequence, that surface-layer turbulence decays when Ri exceeds a critical value, $\mathrm{Ri}_c \approx 0.25$. However, as demonstrated in Sects. 1 and 3 this conclusion is erroneous.

Note that the linear dependences, $\Phi_M \sim \Phi_H \sim z/L$, were traditionally derived from the heuristic "z-less stratification" concept, which postulates that the distance from the surface,

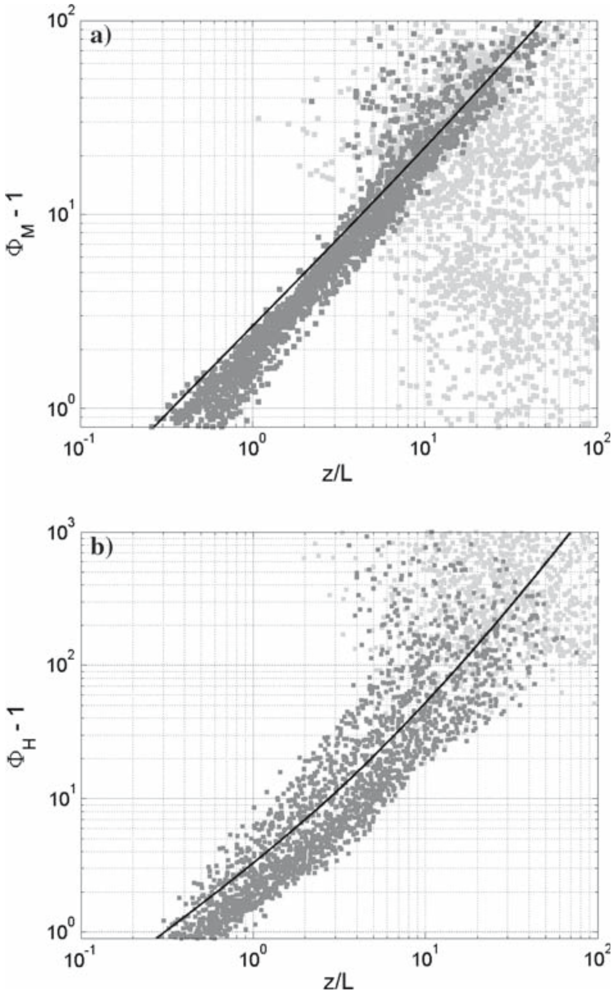

Fig. 6 Dimensionless vertical gradients of (**a**) mean velocity, $\Phi_M = \frac{kz}{\tau^{1/2}}\left(\frac{\partial U}{\partial z}\right)$, and (**b**) potential temperature, $\Phi_H = \frac{k_T z \tau^{1/2}}{-F_z}\left(\frac{\partial \Theta}{\partial z}\right)$, versus z/L, based on our local closure model [solid lines plotted after Eq. (5.13a, b)] compared to the same LES data as in Fig. 5 [from Figs. 1 and 2 of Zilitinkevich and Esau (2007)]

z, does not appear in the set of parameters that characterise the vertical turbulent length scale in sufficiently strong static stability ($z/L \gg 1$). Without this assumption the linear asymptote for Φ_H loses ground while for Φ_M it holds true. Indeed, the existence of a finite upper limit for the flux Richardson number $\mathrm{Ri}_f \to \mathrm{Ri}_f^{\infty}$ at $z/L \to \infty$ immediately yields the asymptotic relation:

$$\Phi_M \approx C_U \frac{z}{L} \text{ at } \frac{z}{L} \gg 1, \tag{68}$$

where $C_U = (\mathrm{Ri}_f^{\infty})^{-1} \approx 5$.

It is important to note that the algebraic closure model presented in Sect. 4, is applicable only to homogeneous turbulence regimes. Therefore it probably serves as a reasonable approximation for the nocturnal ABL separated from the free flow by the neutrally stratified residual layer, but not for the conventionally neutral and the long-lived stable ABLs, which develop against the stably stratified free flow and exhibit essentially non-local features, such as the distant effect of the free-flow stability on the surface-layer turbulence (see Zilitinkevich 2002; Zilitinkevich and Esau 2005). In order to reproduce these types of ABL realistically, an adequate turbulence closure model should take into account the non-local transports.

6 Summary and conclusions

The structure of the most widely used turbulence closure models for neutrally and stably stratified geophysical flows follows Kolmogorov (1941): vertical turbulent fluxes are assumed to be downgradient; the turbulent exchange coefficients, namely, the eddy viscosity, K_M, conductivity, K_H, and diffusivity, K_D, are assumed to be proportional to the turbulent length scale, l_T, and the turbulent velocity scale, u_T, which in turn is taken to be proportional to the square root of the TKE, $E_K^{1/2}$, so that $K_{\{M,H,D\}} \sim E_K^{1/2} l_T$; and E_K is determined solely from the TKE budget equation. Kolmogorov developed this formulation for neutral stratification, where it provides quite a good approximation. However, when applied to essentially stable stratification Kolmogorov's model predicts that TKE decays at Richardson numbers exceeding a critical value, Ri_c (close to 0.25), which contradicts experimental evidence. To avoid this drawback, modern closure models modify the original Kolmogorov formulation assuming $K_{\{M,H,D\}} = f_{\{M,H,D\}}(\mathrm{Ri}) E_K^{1/2} l_z$, where stability functions $f_{\{M,H,D\}}(\mathrm{Ri})$ are determined either theoretically or empirically. Given these functions, it remains to determine l_T and then, apparently, the closure problem is solved.

Such a conclusion, however, is premature. The concepts of the downgradient turbulent transport and the turbulent exchange coefficients, as well as the relationships $K_{\{M,H,D\}} = f_{\{M,H,D\}}(\mathrm{Ri}) E_K^{1/2} l_T$ are consistent with the flux-budget equations only in comparatively simple particular cases relevant to the homogeneous regime of turbulence. Only in these cases the turbulent exchange coefficients can be rigorously defined, in contrast to the turbulent fluxes that represent clearly defined, measurable parameters, governed by the flux-budget equations. It is therefore preferable to rely on the flux-budget equations rather than to formulate hypotheses about virtual exchange coefficients.

Furthermore, the TKE budget equation does not fully characterise turbulent energy transformations, not to mention that the vertical turbulent transports are controlled by the energy of vertical velocity fluctuations, E_z, rather than E_K.

In this study we do not follow the above "main stream" approach, and instead of solely using the TKE budget equation, we employ the budget equations for turbulent potential energy (TPE) and turbulent total energy (TTE = TKE + TPE), which guarantees maintenance of turbulence by the velocity shear in any stratification.

Furthermore, we do not accept *a priori* the concept of downgradient turbulent transports (implying universal existence of turbulent exchange coefficients). Instead, we use the budget equations for key turbulent fluxes and derive (rather than postulate) formulations for the exchange coefficients, when it is physically grounded as in the steady-state homogeneous regime.

In the budget equation for the vertical flux of potential temperature we take into account a crucially important mechanism: generation of the countergradient flux due to the buoyancy effect of potential-temperature fluctuations (compensated, but only partially so, by the correlation between the potential-temperature and the pressure-gradient fluctuations). We show that this is an important mechanism responsible for the principle difference between the heat and the momentum transfer.

To determine the energy of the vertical velocity fluctuations, we modify the traditional return-to-isotropy formulation accounting for the effect of stratification on the redistribution of the TKE between horizontal and vertical velocity components. We then derive a simple algebraic version of an energetically consistent closure model for the steady-state, homogeneous regime, and verify it against available experimental, LES and DNS data.

As seen from Figs. 1–4 showing Ri dependencies of the turbulent Prandtl number, $\mathrm{Pr}_T = K_M/K_H$, the flux Richardson number, Ri_f, dimensionless turbulent fluxes, $(\tau/E_K)^2$ and $F_z^2(E_K E_\theta)^{-1}$, and anisotropy of turbulence, $A_z = E_z/E_K$, our model, in compliance with the majority of data, reveals the existence of two essentially different regimes of turbulence separated by a comparatively narrow interval of Ri around a threshold value of $\mathrm{Ri} \approx 0.25$ (shown in the figures by the vertical dashed lines). On both sides of the transitional interval, $0.1 < \mathrm{Ri} < 1$, the ratios $(\tau/E_K)^2$ and $F_z^2(E_K E_\theta)^{-1}$ approach plateaus corresponding to the very high efficiency of the turbulent transfer at $\mathrm{Ri} < 0.1$, and to the strongly different efficiencies of the momentum transfer (which is still pronounced) and heat transfer (which is very weak) at $\mathrm{Ri} > 1$.

It is hardly incidental that the above threshold coincides with the critical Richardson number, Ri_c, derived from the classical perturbation analyses. These analyses have demonstrated that the infinitesimal perturbations grow exponentially at $\mathrm{Ri} < \mathrm{Ri}_c$ but do not grow at $\mathrm{Ri} > \mathrm{Ri}_c$ when, as we understand now, the onset of turbulent events requires finite perturbations. Consequently, the transitional interval, $0.1 < \mathrm{Ri} < 1$, indeed separates two essentially different regimes: strong turbulence at $\mathrm{Ri} < 0.1$ and weak turbulence at $\mathrm{Ri} > 1$, but do not separate the turbulent and the laminar regimes as traditionally assumed.

What we presented in this study is just a first step towards developing consistent and practically useful turbulence closure models based on a minimal set of equations, which indispensably includes the TTE budget equation and does not imply the existence of the critical Richardson number. Two other recent studies follow this approach. Mauritsen et al. (2007) have developed a simple closure model employing the TTE budget equation and empirical Ri dependences of $(\tau/E_K)^2$ and $F_z^2(E_K E_\theta)^{-1}$ (similar to those shown in our Fig. 2–3). L'vov, Procaccia and Rudenko (Private communication) perform detailed analyses of the budget equations for the Reynolds stresses in the turbulent boundary layer (relevant to the strong turbulence regime) taking into consideration the dissipative effect of the horizontal heat flux explicitly, in contrast to our "effective-dissipation approximation".

As already mentioned, the present study is limited to the local, algebraic closure model that is applicable to the steady-state, homogeneous turbulence regime. Generalised versions of this model, based on the same physical analyses but accounting for the third-order transports (Φ_K, Φ_P, Φ_F and $\Phi_{\{1,2\}}^{(\tau)}$) will be presented in forthcoming papers.

Our data analysis provides only a plausible first verification rather than a comprehensive validation of the proposed model. Special efforts are needed to extend our data analysis using additional field, laboratory and numerically simulated data (e.g., Rohr et al., 1988; Shih et al. 2000). In future work, particular attention should also be paid to direct verification of our approximations, such as those for the term $\rho_0^{-1}\langle\theta\partial p/\partial z\rangle$ taken to be proportional to $\beta\langle\theta^2\rangle$ in Eq. (9b), and for the term $\varepsilon_{i3(\mathrm{eff})} \equiv \varepsilon_{i3}^{(\tau)} - \beta F_i - Q_{i3}$ assumed to be proportional to τ_{i3}/t_T in Eq. (10b).

In the present state, our closure model does not account for vertical transports arising from internal waves. The dual nature of fluctuations representing both turbulence and waves in stratified flows has been suggested, e.g., by Jacobitz et al. (2005). The role of waves and the need for their inclusion in the context of turbulence closure models has been discussed, e.g., by Jin et al. (2003) and Baumert and Peters (2005). Direct account of the wave-driven transports of momentum and both kinetic and potential energies is a topic of our current research.

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Appendix A: The pressure term in the budget equation for turbulent flux of potential temperature

The approximation used in Sect. 2:

$$\beta\left\langle\theta^2\right\rangle+\frac{1}{\rho_0}\left\langle\theta\frac{\partial}{\partial z}p\right\rangle=C_\theta\beta\left\langle\theta^2\right\rangle \tag{69}$$

with C_θ = constant < 1 can be justified as follows. Taking the divergence of the momentum equation we arrive at the following equation

$$\frac{1}{\rho_0}\Delta p=-\beta\frac{\partial}{\partial z}\theta. \tag{70}$$

Applying the inverse Laplacian to Eq. (70) yields

$$\frac{1}{\rho_0}p=\beta\Delta^{-1}\left(\frac{\partial\theta}{\partial z}\right),\quad\text{and}\quad\frac{1}{\rho_0}\left\langle\theta\frac{\partial}{\partial z}p\right\rangle=-\beta\left\langle\theta\Delta^{-1}\frac{\partial^2}{\partial z^2}\theta\right\rangle. \tag{71}$$

We employ the following scaling estimate:

$$\frac{\left\langle\theta\Delta^{-1}\left(\frac{\partial^2\theta}{\partial z^2}\right)\right\rangle}{\left\langle\theta^2\right\rangle}\approx\left(1+\alpha^{-1}\right)\left(1-\frac{\arctan\sqrt{\alpha}}{\sqrt{\alpha}}\right), \tag{72}$$

where $\alpha = l_\perp^2/l_z^2 - 1, l_z$ and $l_\perp$ are the correlation lengths of the correlation function $\langle\theta(t,\mathbf{x}_1)\theta(t,\mathbf{x}_2)\rangle$ in the vertical and the horizontal directions.

Equations (71) and (72) yield

$$\frac{\frac{1}{\rho_0}\left\langle\theta\frac{\partial p}{\partial z}\right\rangle}{\left\langle\theta^2\right\rangle}\approx-\begin{cases}\frac{1}{3}\left(1+\frac{2}{5}\alpha\right) & \text{in the thermal isotropy limit }(\alpha\ll 1)\\ 1-\frac{\pi}{2\sqrt{\alpha}} & \text{in the infinite thermal anisotropy limit }(\alpha\gg 1).\end{cases} \tag{73}$$

Consequently, the coefficient C_θ= {1 + [r.h.s. of Eq. (73)]} turns into 2/3 in the thermal isotropy limit (corresponding to neutral stratification) and vanishes in an imaginary case of the infinite thermal anisotropy. Our empirical estimate, Eq. (63), of C_θ= 0.3 is a reasonable compromise between these two extremes.

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