Poisson line processes

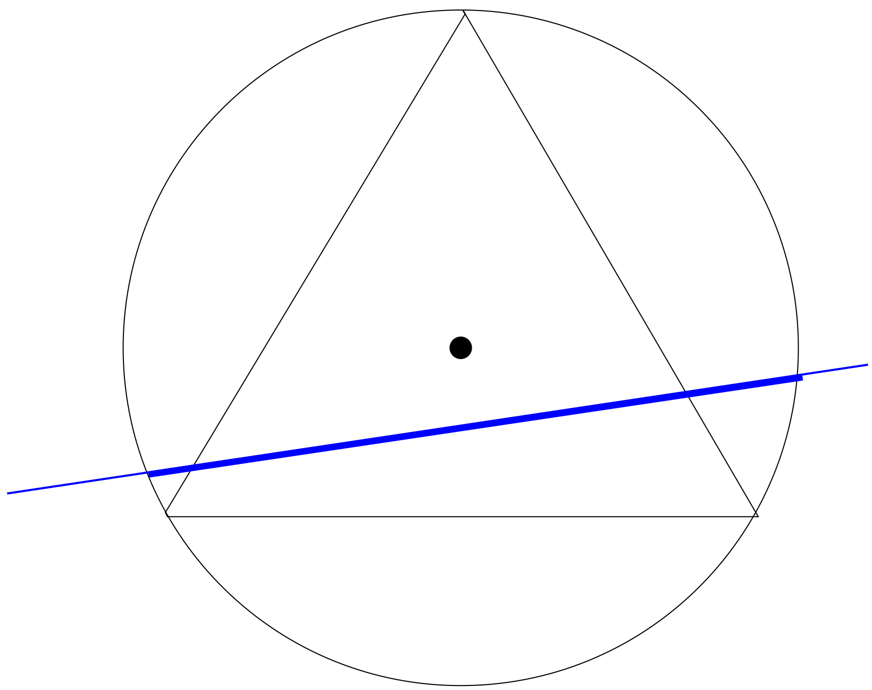
C. Lantuéjoul

MinesParisTech
christian.lantuejoul@mines-paristech.fr

Bertrand paradox

A problem of geometrical probability

A line is thrown "at random" on a circle. What is the probability that the induced chord exceeds the side of the inscribed equilateral triangle?

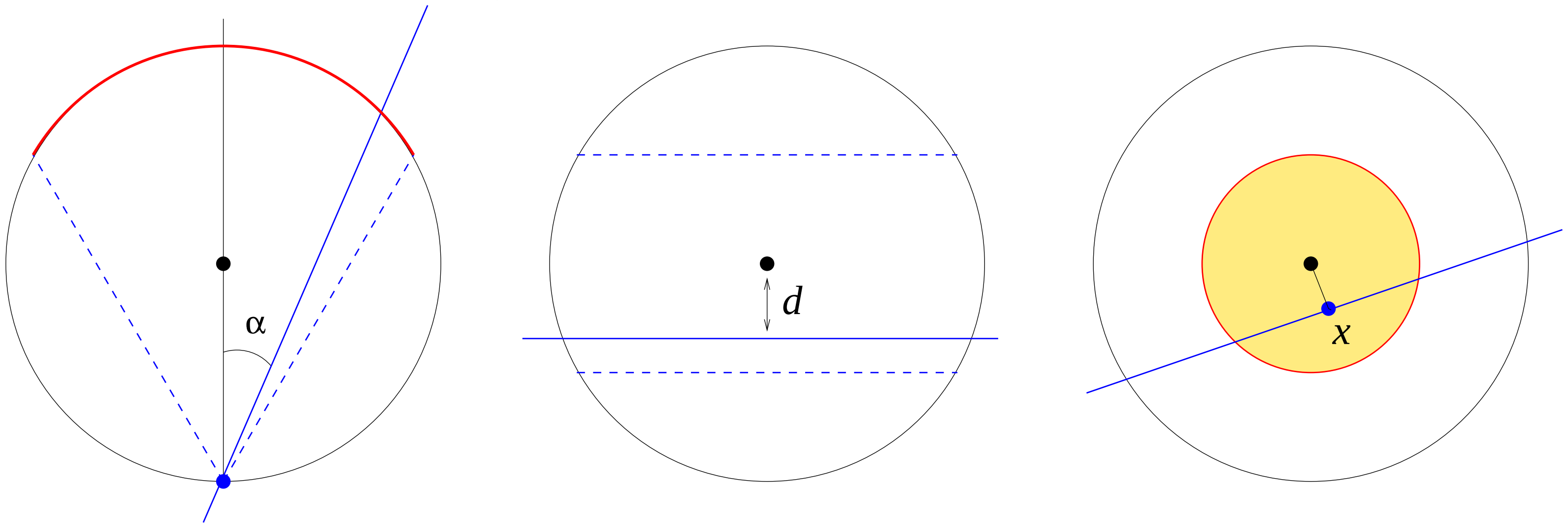


What do we mean by random?

Model 1: The chord endpoints are independent and uniform on the circle. Then the probability is $1/3$;

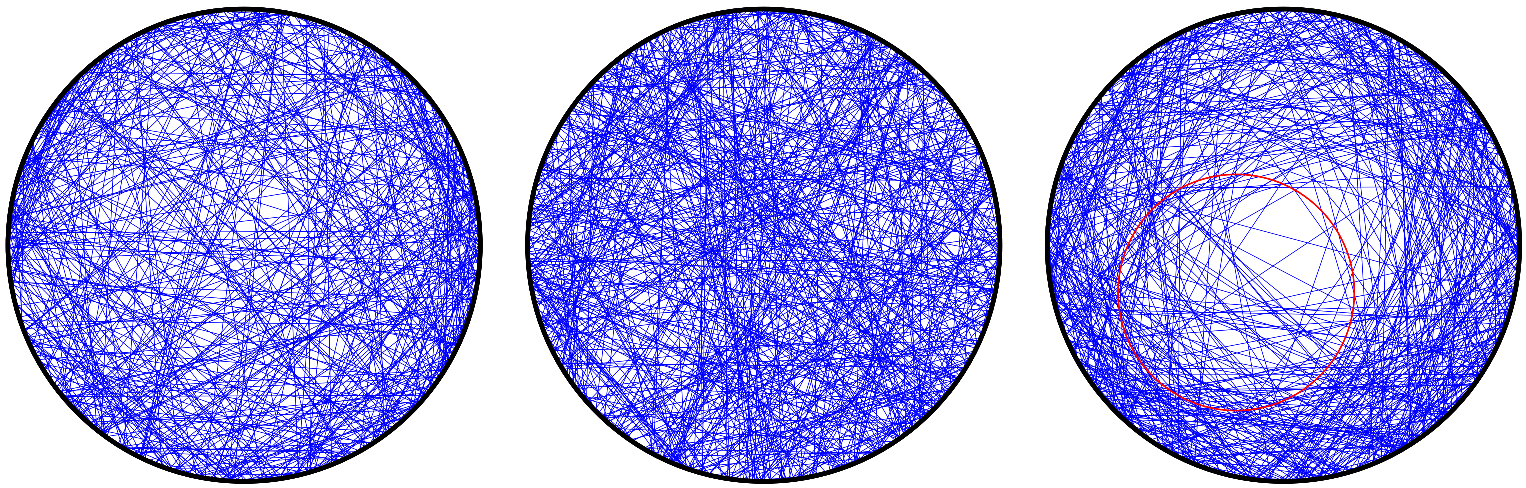
Model 2: The distance from the centre to the line is uniform. Then the probability is $1/2$;

Model 3: The projection of the centre onto the line is uniform over the disc. Then the probability is $1/4$.



Remarks

– A small simulation exercise shows that the three models provide different chord distributions.



– Bertrand's problem is raised irrespective of the size or the location of the disk considered. Jaynes (1973) argues that the searched probability should remain the same whenever the disc is replaced by a smaller one. This is only the case for model 2.

Outline

Poisson lines

– definition

– number of lines hitting a domain

– orientations

Poisson polygons

– fundamental and typical polygons

– conditional invariance by erosion

Applications

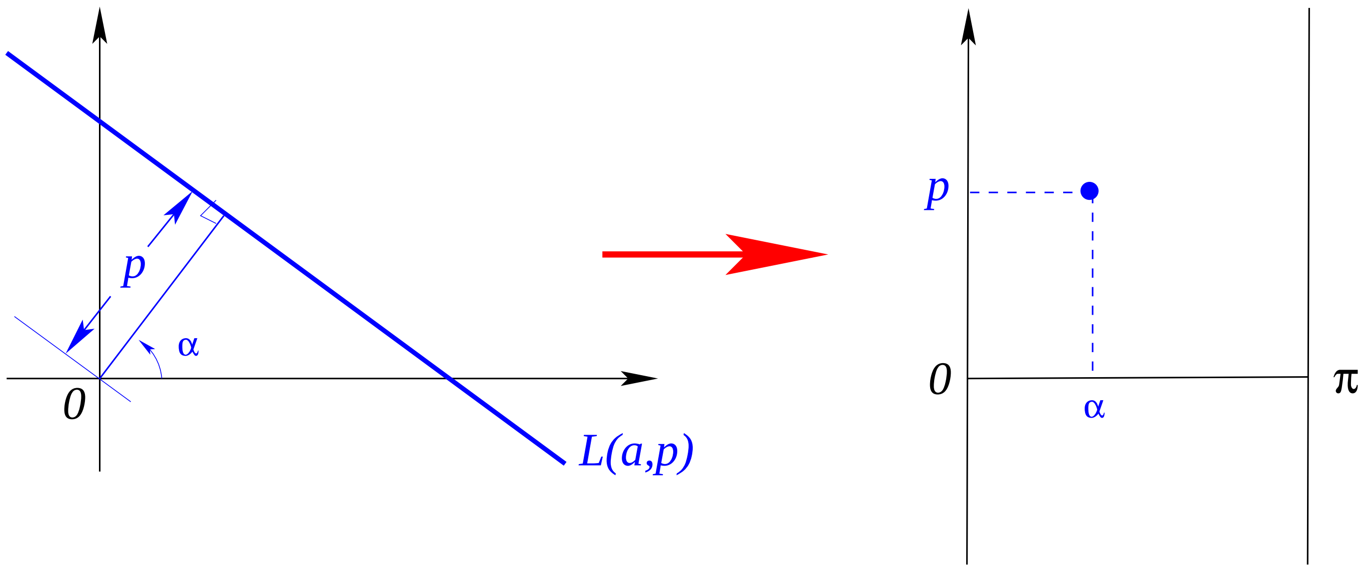
– STIT tessellation

– Chentsov's model

– substitution random functions

Poisson lines

Parametrization of a line

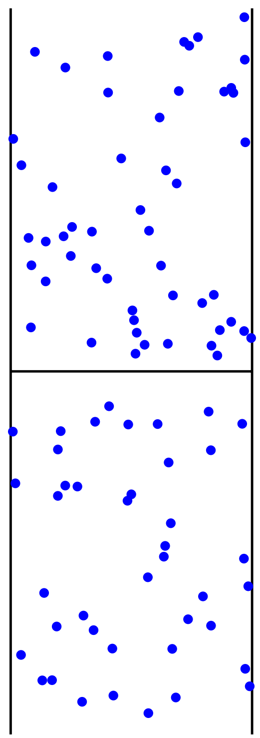


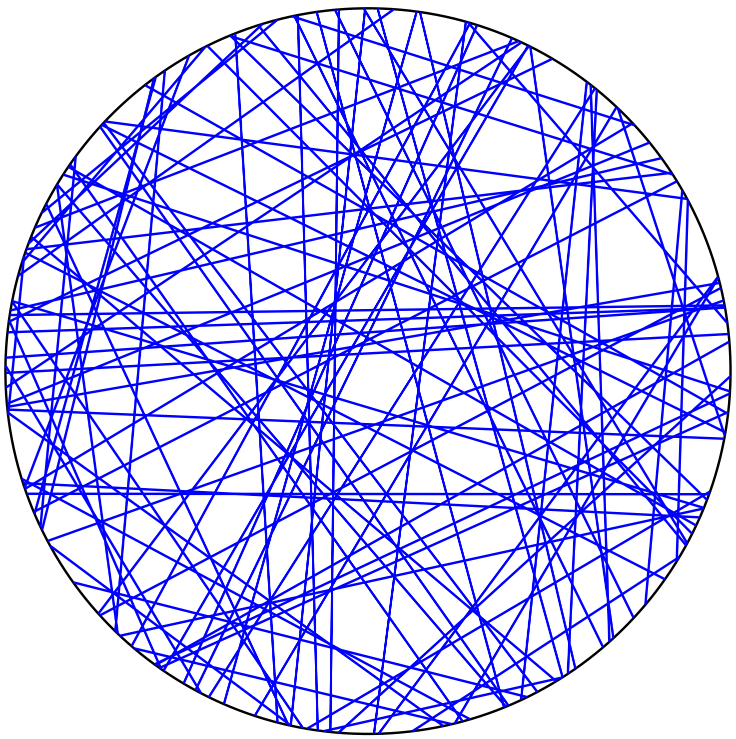
Equation of a line:

$$x \cos \alpha + y \sin \alpha = p$$

$0 \leq \alpha < \pi$ direction $\qquad -\infty < p < +\infty$ location

Poisson line process

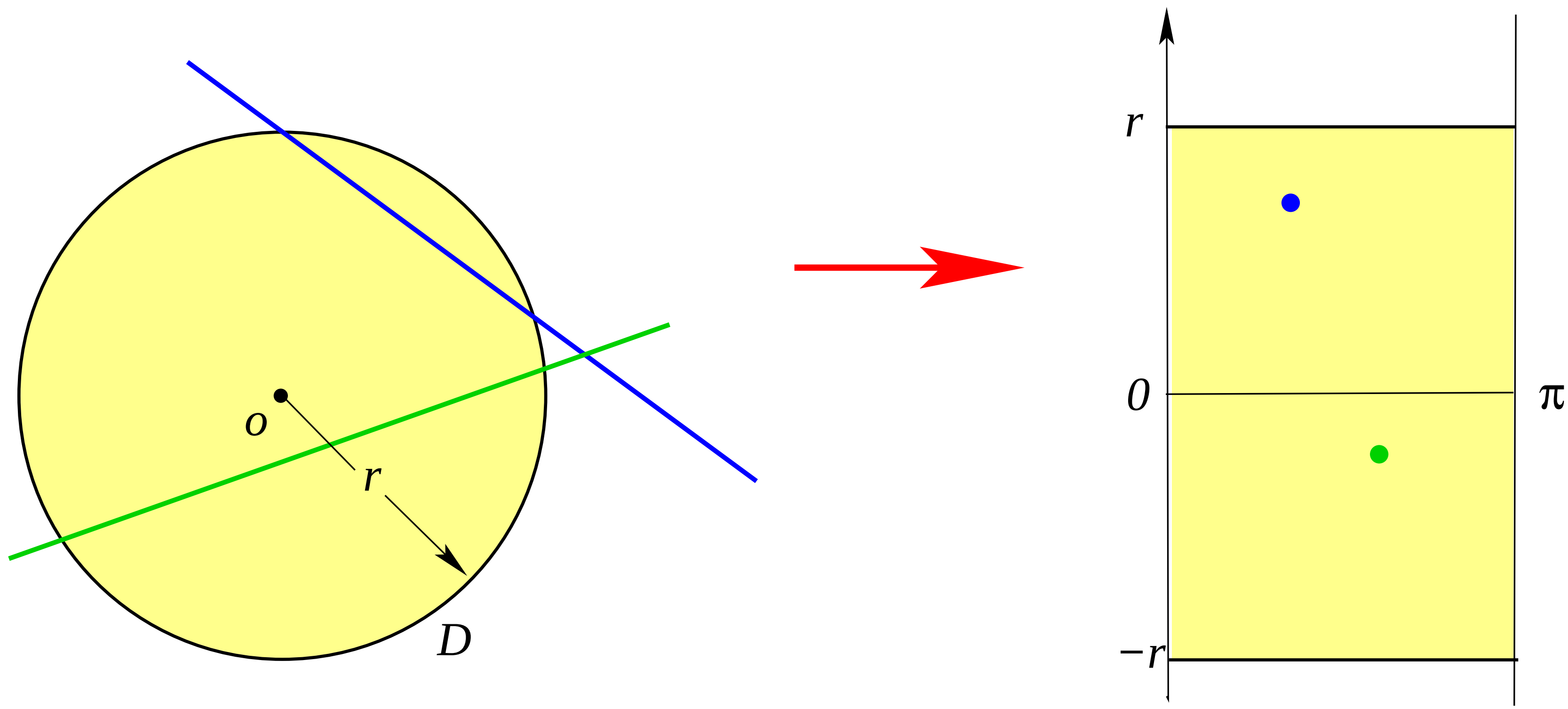




Definition:

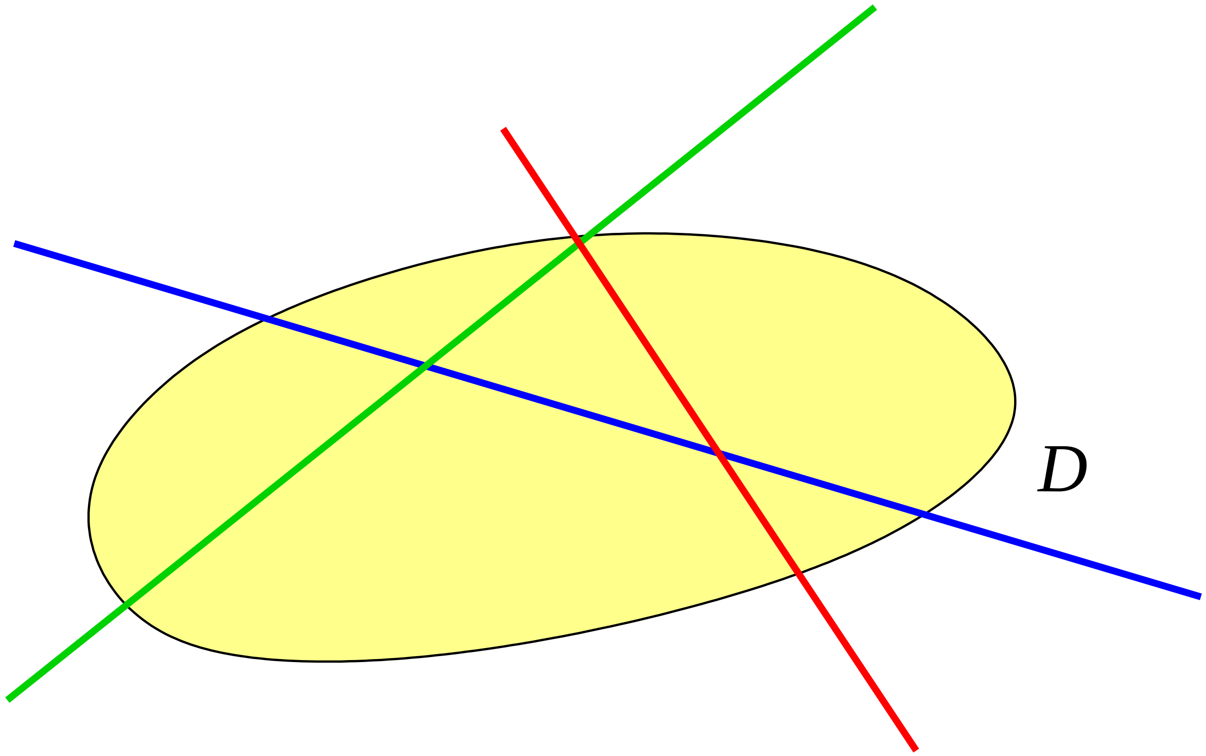
A Poisson line process with intensity λ is parametrized by a Poisson point process with intensity λ on $[0, \pi[\times \mathbb{R}$.

Number of Poisson lines hitting a disk



The number of lines hitting the disk $D(o, r)$ follows a Poisson distribution with mean $\lambda \times \pi \times 2r = \lambda p\big(D(o, r)\big)$

Number of Poisson lines hitting a convex domain

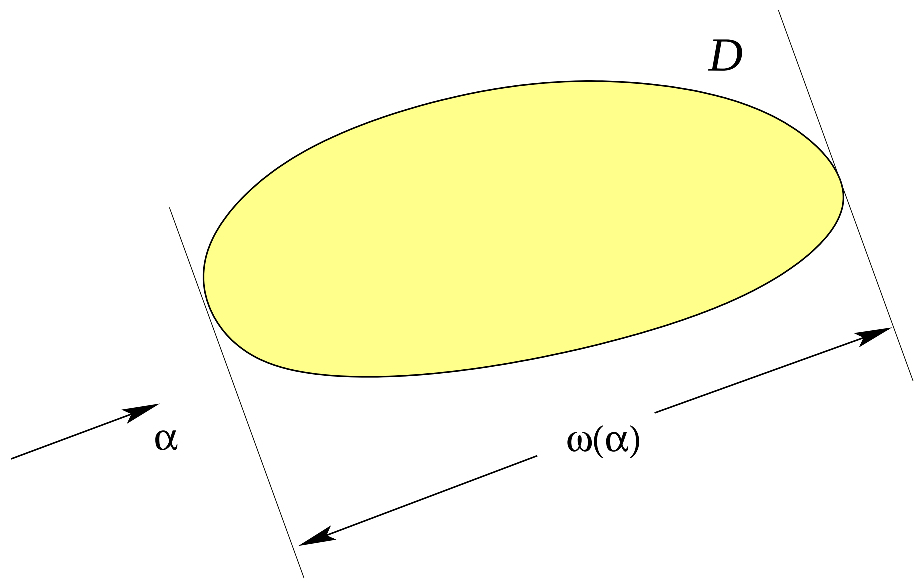


The number of lines hitting a convex domain D follows a Poisson distribution with mean $\lambda p(D)$

($p(D) =$ perimeter of D)

Uniform line hitting a convex domain

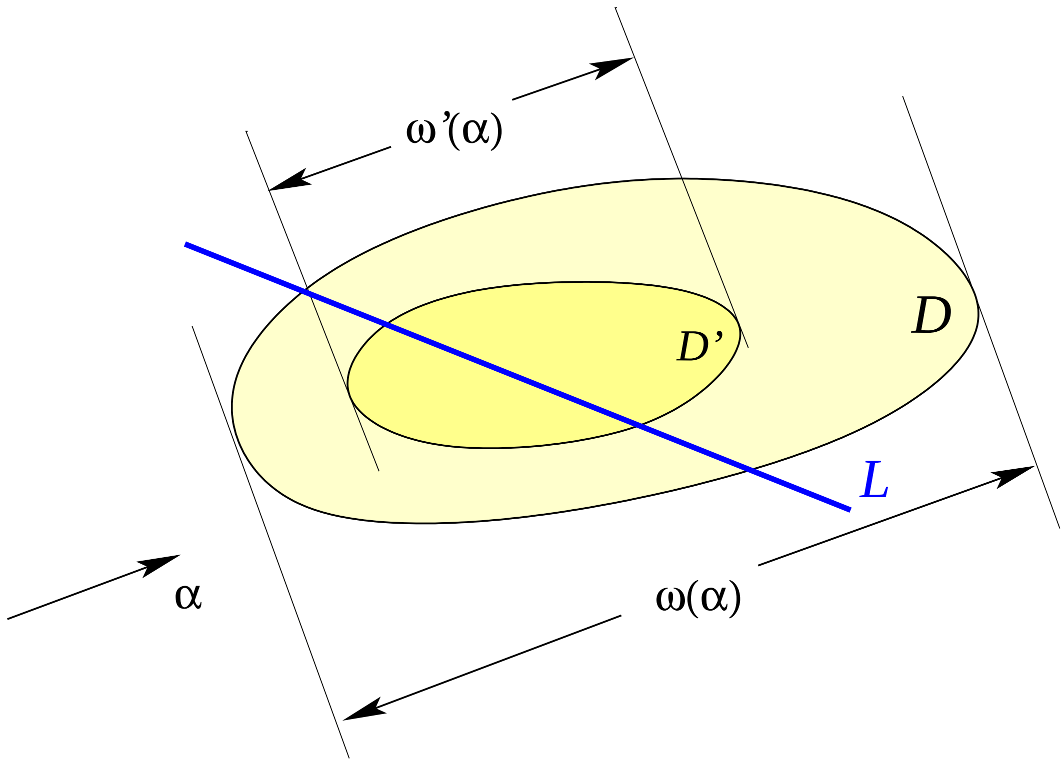
Each Poisson line hitting D derives from a point uniformly located on $\ell(D) \subset [0, \pi[\times \mathbb{R}$. The Poisson line is called a uniform line hitting D.



The direction of a uniform line hitting D is not uniform

$$f(\alpha) = \frac{\omega(\alpha)}{\int_0^{\pi} \omega(\alpha)\, d\alpha} = \frac{\omega(\alpha)}{p(D)}$$

Uniform line hitting a convex domain (2)

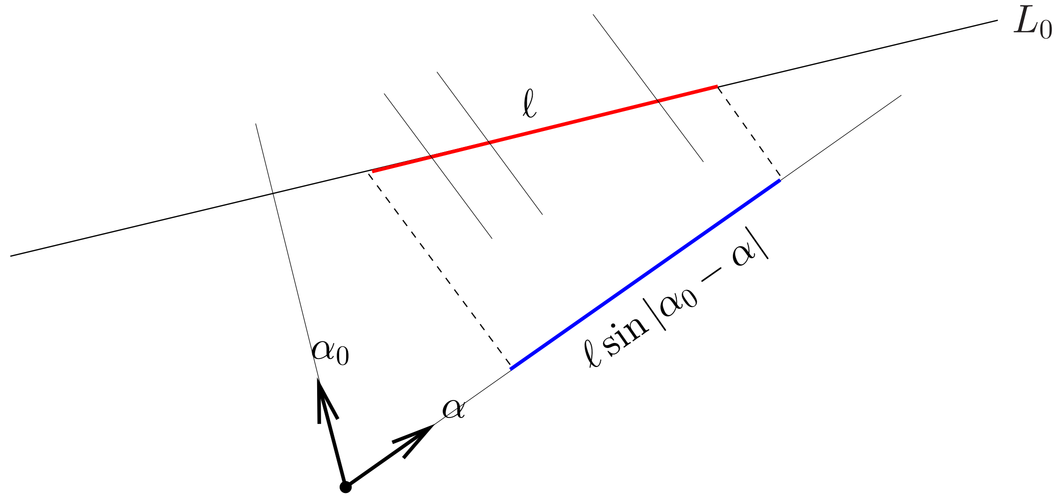


$$P\{L \cap D' \neq \emptyset \,|\, L \cap D \neq \emptyset\} = \int_0^{\pi} \frac{\omega'(\alpha)}{\omega(\alpha)} \frac{\omega(\alpha)}{p(D)} \, d\alpha = \frac{p(D')}{p(D)}$$

Directions of Poisson lines hitting a fixed line

Remark:

Even if the Poisson line network is isotropic, the distribution of the angles between Poisson lines and a fixed line is not uniform on $]0,\pi[$.



Property:

The directions of the Poisson lines hitting L_0 follow the pdf

$$f(\alpha) = \frac{\ell \sin|\alpha_0 - \alpha|}{\int_0^\pi \ell \sin|\alpha_0 - \alpha| d\alpha} = \frac{\sin|\alpha_0 - \alpha|}{2} \qquad 0 \le \alpha \le \pi$$

Poisson polygons

Poisson polygon

Definition:

Poisson polygons are delimited by Poisson lines.

Statistical properties:

There are two distinct ways of specifying the distribution associated with some attribute (area, perimeter, number of vertices) of Poisson polygons:

– assign the same weight to each polygon of one realization. The statistical properties obtained are those of a virtual polygon, called the typical polygon.

– consider the statistical properties of the polygon that contains a fixed point, for instance the origin (fundamental polygon).

Result: (Miles, 1969)

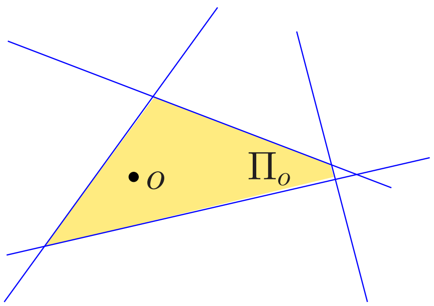
The distributions of the typical polygon and of the fundamental polygon are fairly different:

– those of the typical polygon are weighted in number;

– those of the fundamental polygon are weighted in area.

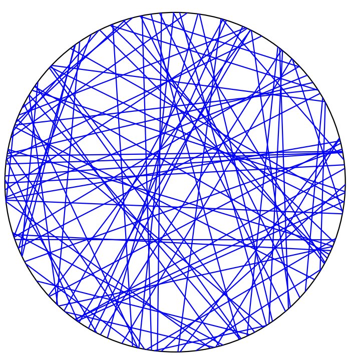
Example: mean area of a polygon

Fundamental polygon:



$$E\{a(\Pi_o)\} = \int_{\mathbb{R}^2} P\{x \in \Pi_o\}\, dx$$
$$= \int_{\mathbb{R}^2} e^{-2\lambda|x|}\, dx = \frac{\pi}{2\lambda^2} \approx \frac{1.57}{\lambda^2}$$

Typical polygon:



$$E\{a(\Pi)\} = \lim_{r \longrightarrow \infty} \frac{\pi r^2}{N(r)} = \frac{1}{\pi\lambda^2} \approx \frac{0.32}{\lambda^2}$$

$N(r)$ = number of polygons contained in a disk of radius r

On distributions weighted in number and in area

Result: (Miles, 1969)

If ψ stands for some attribute of a polygon with area a, then the distributions weighted in number F and in area F_o are related by the formulae

$$dF_o(\psi, a) = \frac{a\, dF(\psi, a)}{E\{a\}} \qquad dF(\psi, a) = \frac{a^{-1}\, dF_o(\psi, a)}{E_o\{a^{-1}\}}$$

Compendium of expectations: (Miles, 1969; Matheron, 1975)

	Typical	Fundamental
Area	$\frac{1}{\pi\lambda^2} \approx \frac{0.32}{\lambda^2}$	$\frac{\pi}{2\lambda^2} \approx \frac{1.57}{\lambda^2}$
Perimeter	$\frac{2}{\lambda}$	$\frac{\pi^2}{2\lambda} \approx \frac{4.93}{\lambda}$
Number of vertices	4	$\frac{\pi^2}{2} \approx 4.93$

STIT model

STIT model

Introduction:

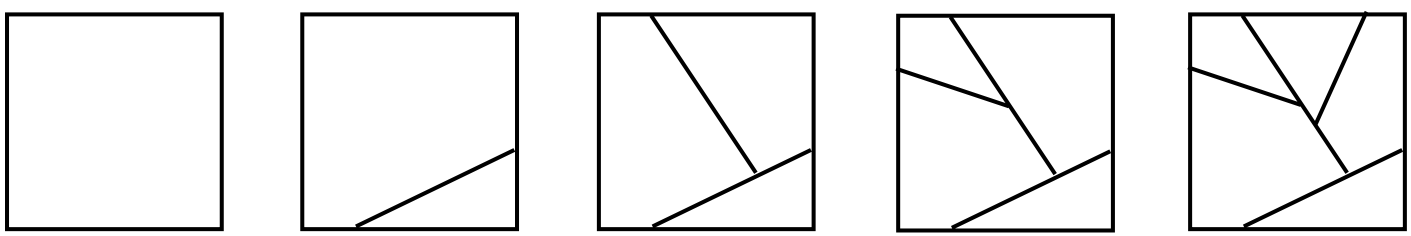
This iterative model (STIT stands for STable by ITeration) was designed by Nagel and Weiss for nested tessellations, apt to represent crack or fracture structures.

Main features:

This model is indexed by time. At each time, it consists of a tessellation of polygonal fragments.

Each fragment remains intact during an exponential time, the mean of which is inversely proportional to its perimeter.

After that time, a uniform line splits it into two fragments.



STIT algorithm

Objective:

Generate the STIT model in the polygonal domain X.

Algorithm:

(i) set $\mathcal{P} = \{X\}$, and generate $\tau_X = \varepsilon_{p(X)}$;

(ii) find the polygon $P \in \mathcal{P}$ that minimizes τ_P;

(iii) generate a uniform line that splits P into two polygons Q and R;

(iv) generate $\tau_Q = \tau_P + \varepsilon_{p(Q)}$ and $\tau_R = \tau_P + \varepsilon_{p(R)}$;

(v) remove P from $\mathcal{P}$, insert Q and R to $\mathcal{P}$, and goto (ii).

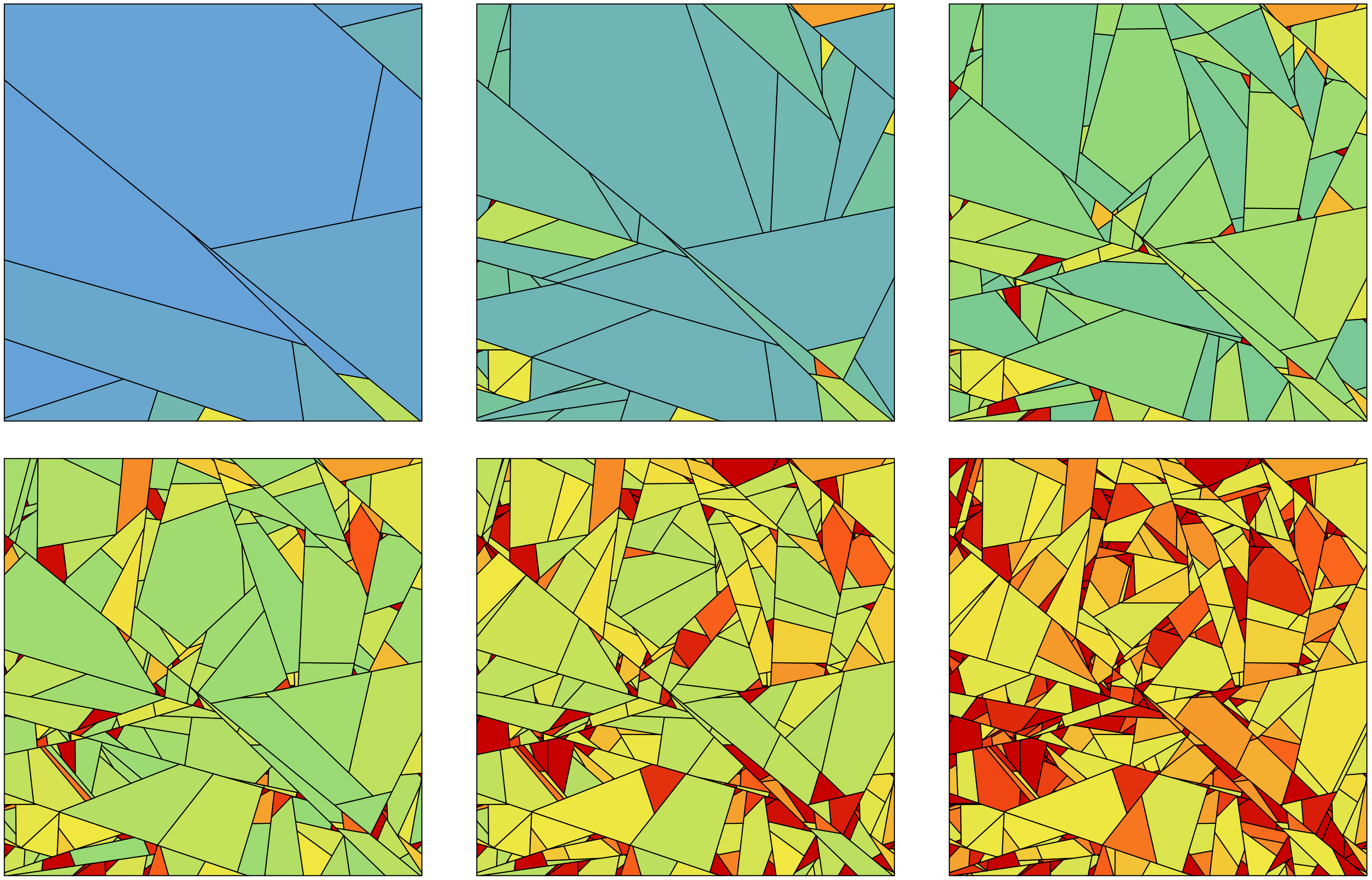
Used notation:

$\mathcal{P}$ is a population of polygons;

ε_a is an exponentiel variable with parameter a, or mean $\frac{1}{a}$;

$p(P)$ is the perimeter of P.

Illustrating the STIT model



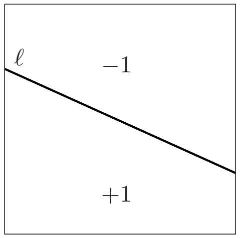
Here $X = [-1, +1]^2$. The simulation is depicted at times $1, 2, 3, 4, 5$ and 6.

Chentsov's model

Chentsov's model

Definition

Consider a network $\mathcal{L}$ of Poisson lines with intensity λ. Each line splits the plane in two parts. One part chosen at random is set to $+1$, the other one to -1. This defines a random function V_ℓ that looks as below:



We proceed the same way for each Poisson line. We finally put

$$Z(x) = \sum_{\ell \in \mathcal{L}} [V_\ell(x) - V_\ell(o)] \qquad x \in \mathbb{R}^2$$

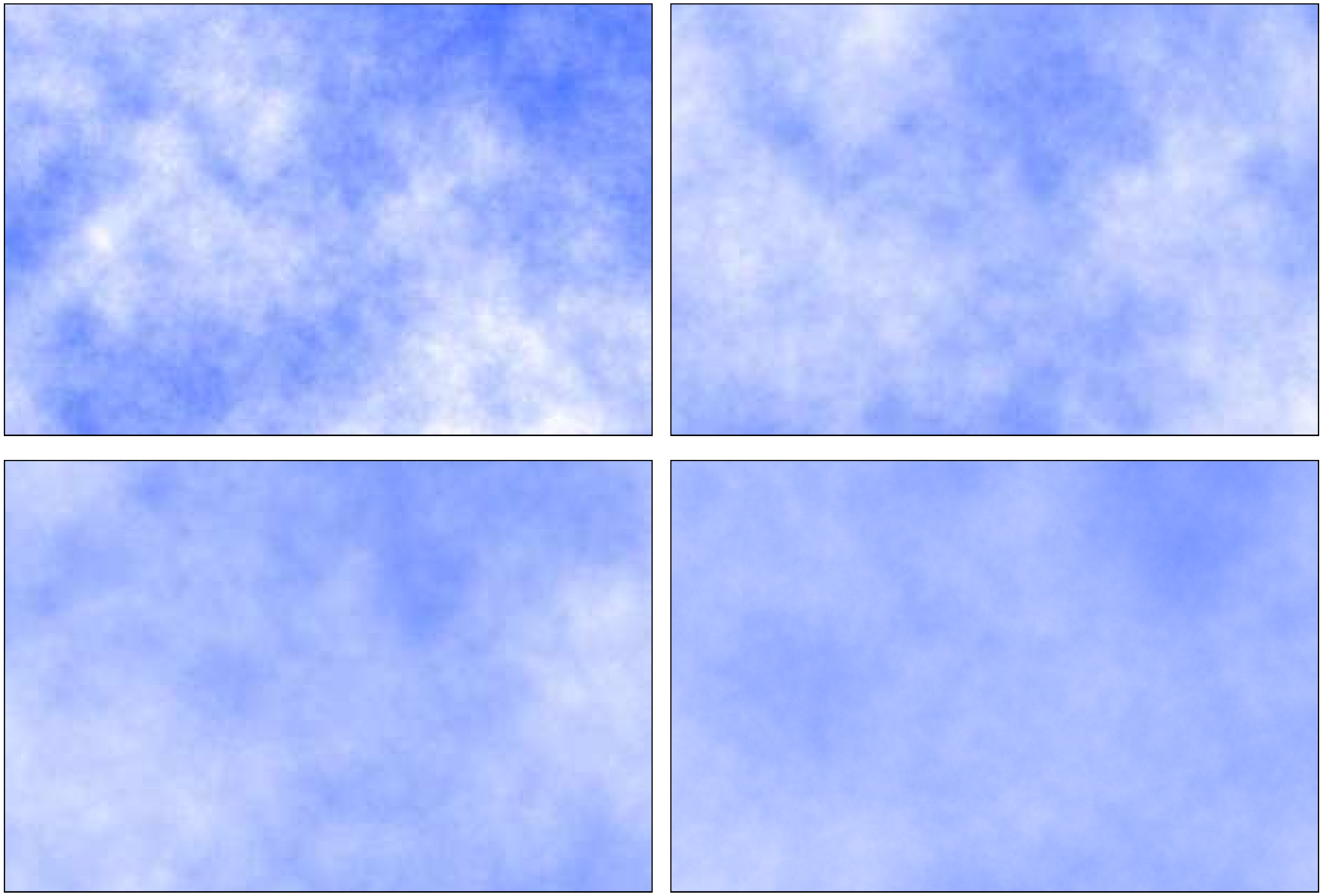
where o is the origin of $\mathbb{R}^2$.

Remark:

$Z(x)$ depends only on the lines hitting $]o, x[$. In particular $Z(o) = 0$.

Chentsov's model (2)

Simulation at 4 different scales:



Chentsov's model (3)

Structure:

Z is a random function with stationary increments:

$$Z(x) - Z(y) = \sum_{\ell \cap]x,y[\neq \emptyset} [V_\ell(x) - V_\ell(y)]$$

$$\stackrel{\mathcal{D}}{\equiv} \sum_{\ell \cap]x-y,o[\neq \emptyset} [V_\ell(x-y) - V_\ell(o)]$$

$$= Z(x-y)$$

Variogram:

$$\frac{1}{2} Var\{Z(x) - Z(y)\} = 4\lambda |x-y| \qquad x, y \in I\!R^2$$

Dispersion variance:

$$\sigma^2(o, D) = 4\lambda E\{|X-Y|\} \qquad X, Y \sim \mathcal{U}(D)$$

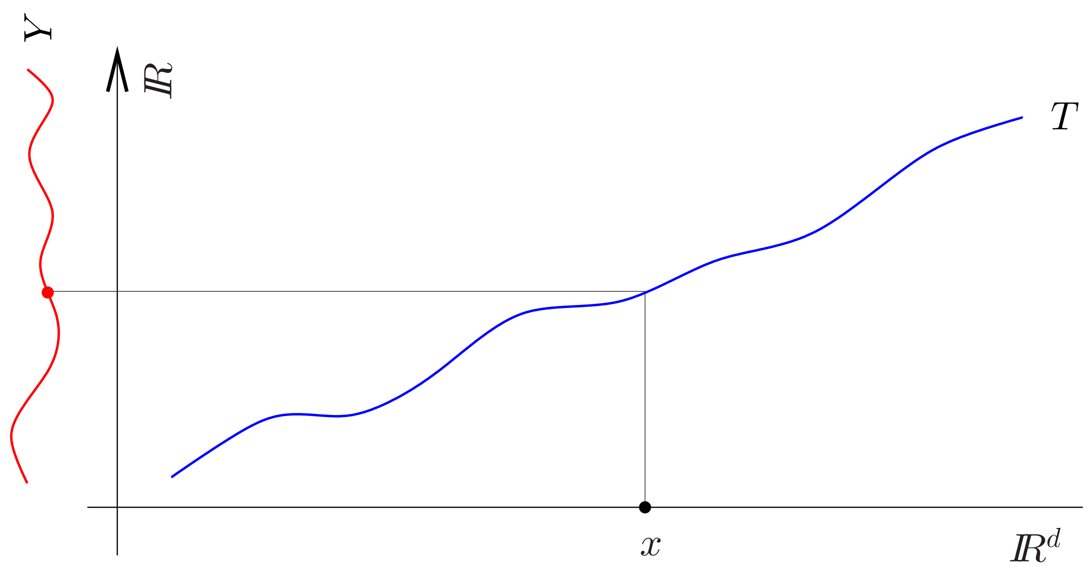
Substitution random functions

Substitution random function

Definition:

A substitution random function is a composition of a random function with random increments directing function and a stationary process coding process:

$$Z(x) = Y \circ T(x) \qquad x \in \mathbb{R}^d$$



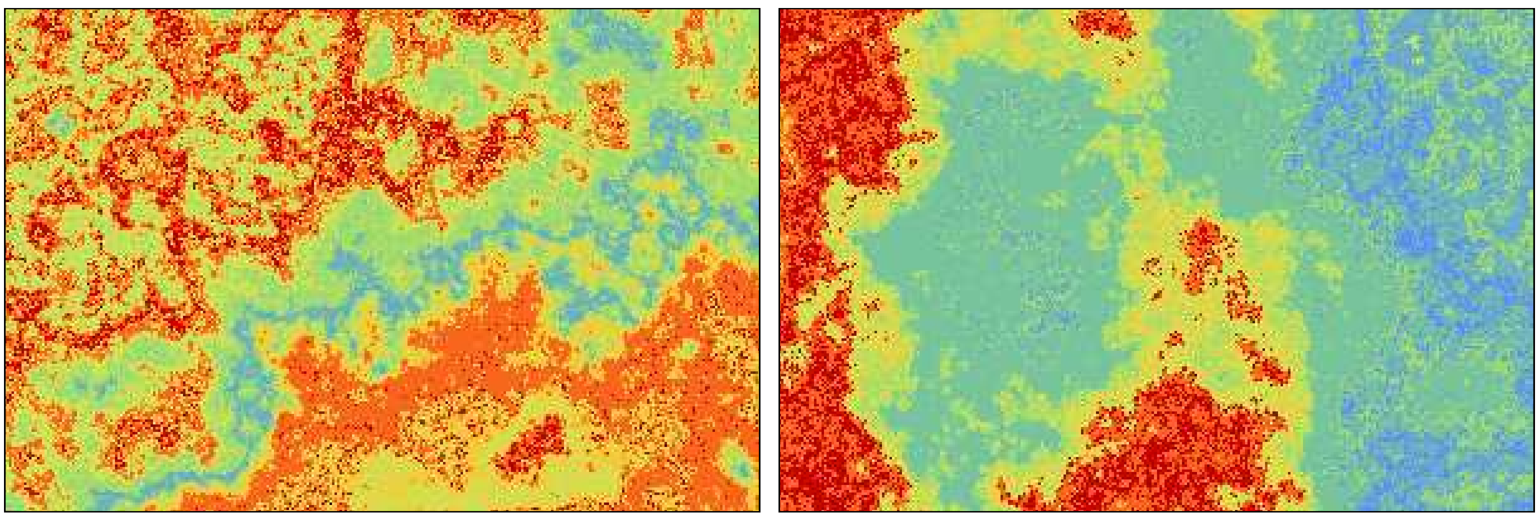
Substitution random function (2)

Ingredients:

– the directing process is a Chentsov's model;

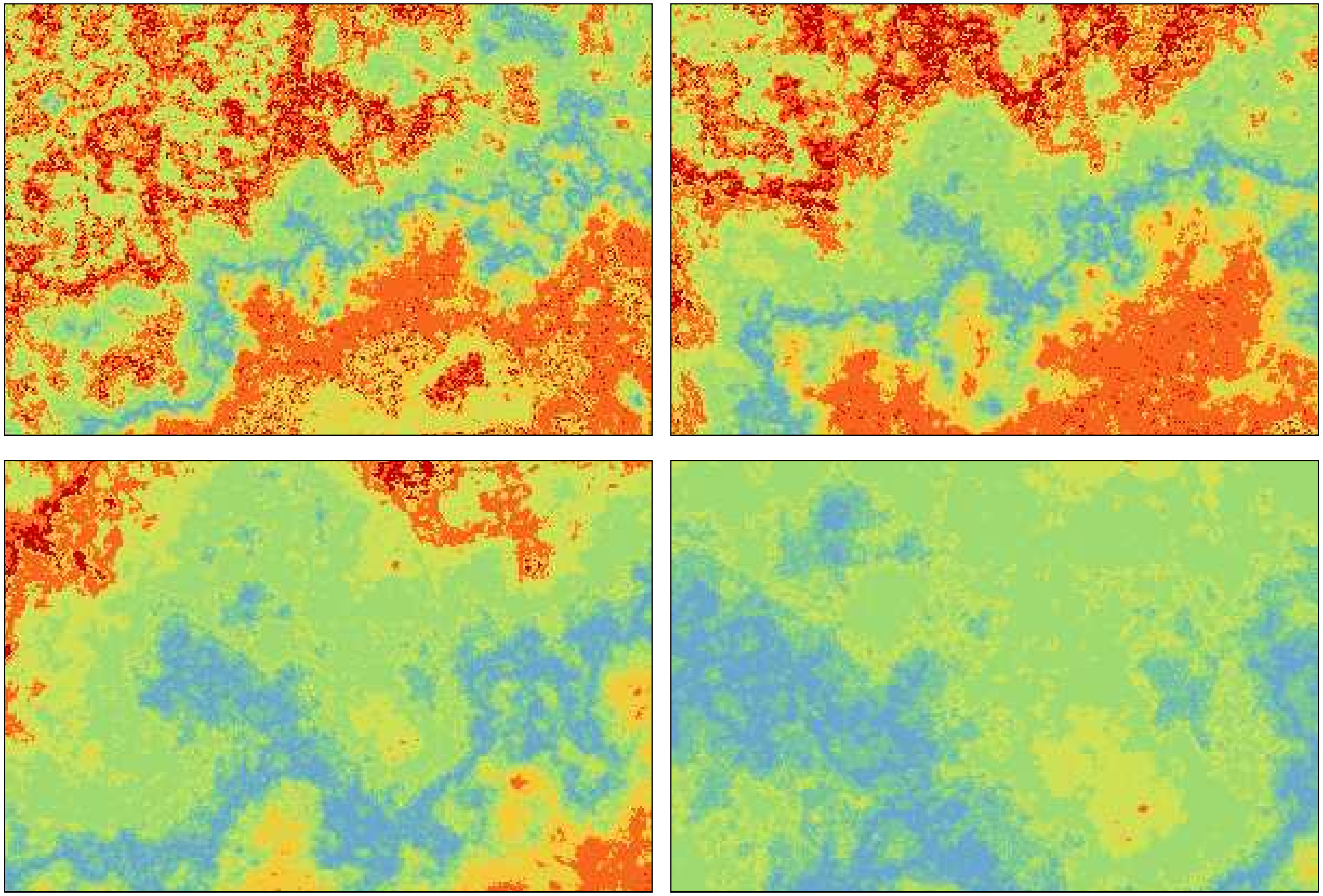
– the coding process is a Markov chain with a uniform stationary distribution (8 states)

Examples:



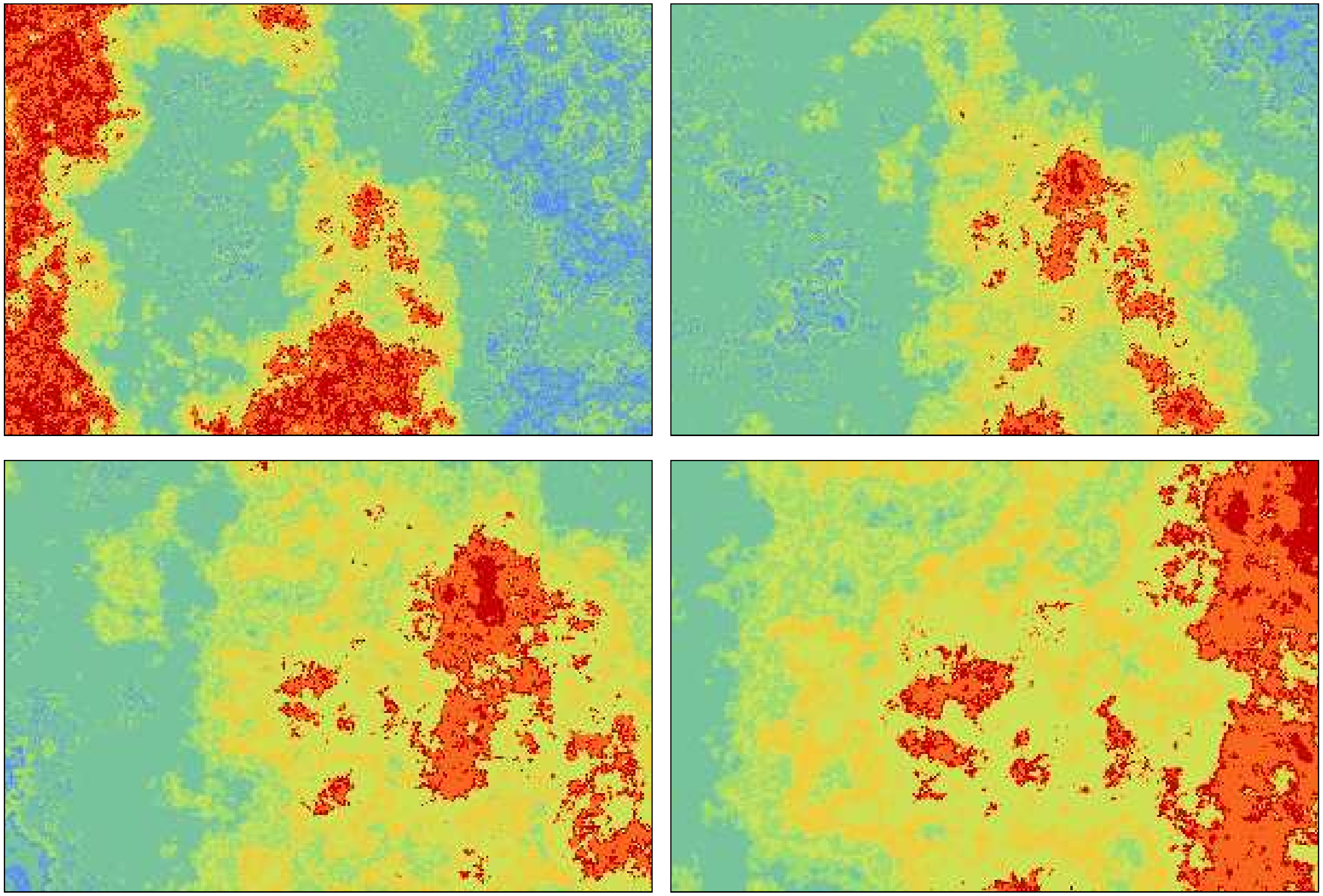
Substitution random function (3)

Simulation at 4 different scales:



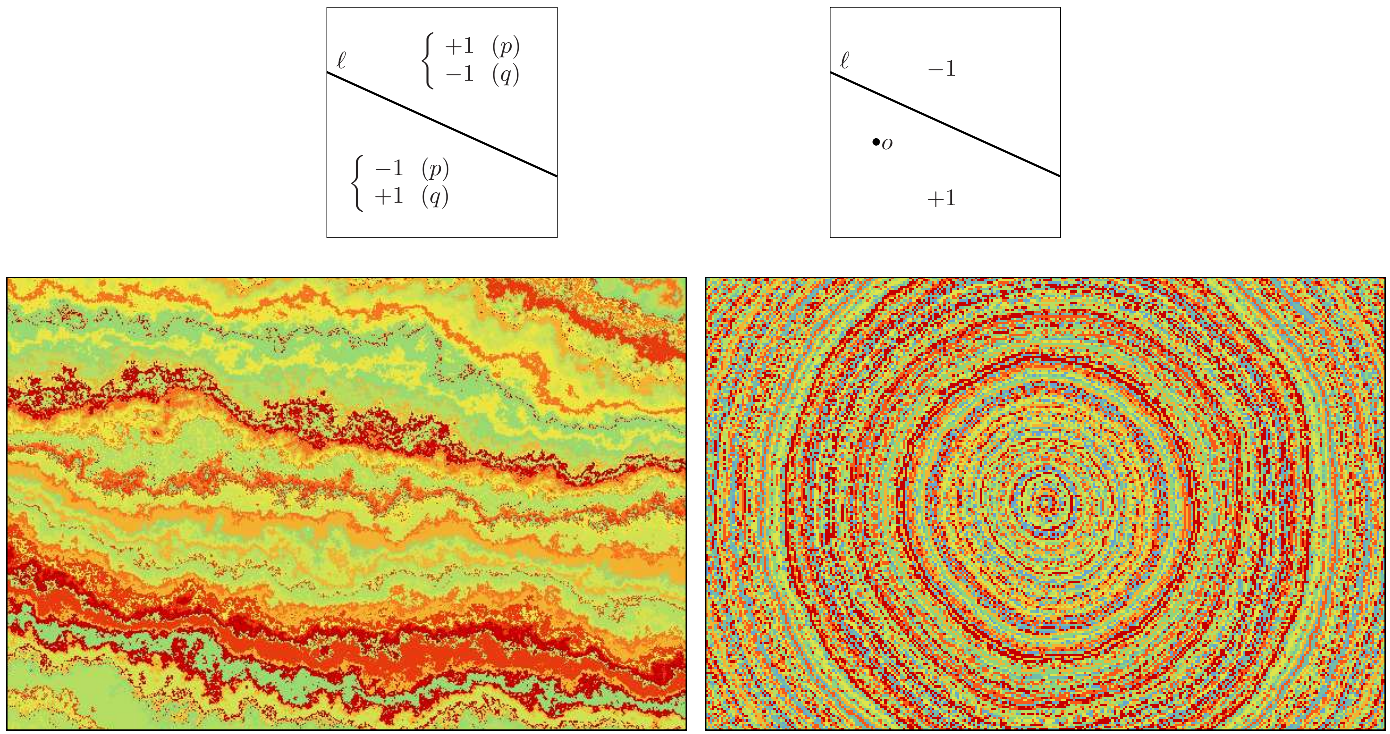
Substitution random function (4)

Simulation at 4 different scales:



Substitution random function (5)

The geometry of the realizations is governed by the directing function:



The topology of the realizations are governed by the coding process ... but this is another story!

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